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Disordered lattice networks: general theory and simulations

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SUMMARY

In this work we develop a theory for describing random networks of resistors of the most general topology. This approach generalizes and unifies several statistical theories available in literature. We consider an *n*-dimensional anisotropic random lattice where each node of the network is connected to a reference node through a given random resistor. This topology includes many structures of great interest both for theoretical and practical applications. For example, the one-dimensional systems correspond to random ladder networks, two-dimensional structures model films deposited on substrates and three-dimensional lattices describe random heterogeneous materials. Moreover, the theory is able to take into account the anisotropic percolation problem for two- and three-dimensional structures. The analytical results allow us to obtain the average behaviour of such networks, i.e. the electrical characterization of the corresponding physical systems. This effective medium theory is developed starting from the properties of the lattice Green's function of the network and from an *ad hoc* mean field procedure. An accurate analytical study of the related lattice Green's functions has been conducted obtaining many closed form results expressed in terms of elliptic integrals. All the theoretical results have been verified by means of numerical Monte-Carlo simulations obtaining a remarkably good agreement between numerical and theoretical values. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: random networks; Green functions; mean field theory

INTRODUCTION

Disordered resistors networks have been, for many years, very useful tools to model transport phenomena in heterogeneous or composite physical systems. The first studies, from a general point of view, were developed by Kirkpatrick in the context of the isotropic transport and the percolation in random lattice [1, 2]. In these works, the theoretical description of conduction was provided by a so-called effective medium theory. This theory, originally formulated to

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describe the conductivity of binary mixture [3, 4], has been extended and adapted to treat disordered networks. Moreover, some attempts to generalize the theory to anisotropic random networks were made to clarify some general aspects of conduction in anisotropic materials [5]. During the evolution of such theories many approaches have been used to obtain statistical information about the behaviour of heterogeneous systems. In Reference [6] exact fields calculations lead to exact effective properties in some particular cases; moreover, alternative theoretical circuit theory approaches have been adopted to obtain the electrical properties of statistical mixtures [7]. More recently, the theories for linear random networks have been generalized to the case of non-linear random networks obtaining the equivalent non-linear behaviour of such heterogeneous systems [8, 9].

In all these cases the most considered topology is the simple two- or three-dimensional grid of resistors, which mimics the heterogeneity in two- or three-dimensional composite materials. In this work we devote our attention to a generalized topology where each node of the grid is connected to another external node (called substrate in the following) by means of a given resistors. In Figure 1 one can find an example of such network for two-dimensional systems. The random character of the network may regard the resistors in the grid or the resistors towards the substrate or in the more complex case both of them. So, to obtain a general theory we develop our calculations for an arbitrary n-dimensional lattice with substrate. Moreover, the statistical distribution of the resistor values in the grid may follow different probability laws in the different directions of the grid. This possibility allows us to describe anisotropy of the system. In actual applications the most important particular cases for such a topology are the following: (i) the one-dimensional case, which corresponds to a classical ladder network; (ii) the two-dimensional anisotropic or isotropic case with or without substrate (useful to

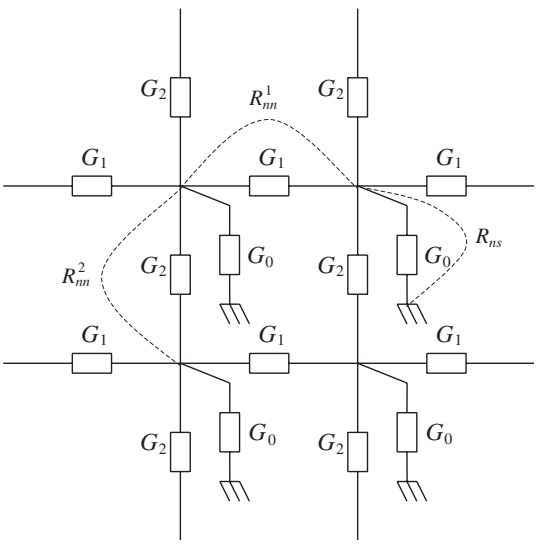

Figure 1. Two-dimensional version of the general topology network used in this work. One can observe the anisotropy of the network and the definition of the characteristic node–node and node–substrate resistance.

describe two-dimensional mixtures or films deposited on substrates with electric interaction); and (iii) three-dimensional isotropic, transversely isotropic or completely anisotropic networks without substrate (adopted to study random heterogeneous materials).

The method applied to develop the general theory is based on two main steps: firstly an analysis of the homogeneous networks with substrate based on the lattice Green's functions is performed. This approach [10–13] permits to obtain exact results about the electrical behaviour of infinite regular lattice networks. The second step consists in applying an *ad hoc* averaging procedure based on the effective medium theory. This approach is a generalization of the standard one in order to take into account the electrical interaction with the substrate. The application of the two steps allows us to find a strong conceptual connection between the lattice Green's function of the network and the problem of obtaining the average behaviour of random grids. An accurate analytical study of the related lattice Green's functions has been conducted obtaining many closed form results expressed in terms of elliptic integrals. All the theoretical results have been verified by means of numerical Monte-Carlo simulations obtaining a remarkably good agreement between numerical and theoretical values.

LATTICE RESISTANCE FUNCTIONS

To approach the problem of the general disordered resistance networks it is important to known, as preliminary information, the resistive behaviour of homogeneous infinite lattice networks. We shall be concerned with disordered network models of the most general type. Therefore, we start by analysing the following general lattice topology: we take into consideration an anisotropic n-dimensional homogeneous grid. It means that we define different values for the conductances aligned along the different lattice directions. Homogeneous grid means that all the conductances in a given direction have the same value. In particular, the value of all the conductances in direction h ($h = 1,\ldots,n$) will be indicated with G_h. Moreover, each node of the grid is connected with a substrate node (another external node not belonging to the grid) through a conductance G_0. For example, a two-dimensional version of the homogeneous network with substrate is represented in Figure 1. Generally speaking, each node is connected to $2n + 1$ resistors: two resistors for each spatial lattice direction (one before and one after the node) and one resistor (G_0) towards the substrate. We represent the position of a given node with the integer co-ordinates $\bar{x} \in Z^n$, and we consider the associated electrical potential indicated as $V(\bar{x})\ \forall \bar{x} \in Z^n$.

To characterize such a network we take into consideration two arbitrary nodes i and j (represented by the lattice positions $\bar{x}^i$ and $\bar{x}^j$) and the common node 0 of the substrate. We suppose that two given currents I_i and I_j flow in terminal connected to the nodes $\bar{x}^i$ and $\bar{x}^j$, in order to define a two-port network (see Figure 2 for details). The current Kirchhoff law applied to the generic node $\bar{x}$ reads:

$$\sum_{k=1}^{n} G_k[2V(\bar{x}) - V(\bar{x} + \bar{e}_k) - V(\bar{x} - \bar{e}_k)] + G_0 V(\bar{x}) = I_i \delta(\bar{x},\bar{x}^i) + I_j \delta(\bar{x},\bar{x}^j) \tag{1}$$

where $\delta(\bar{x}, \bar{y})$ is the Kronecker's delta function ($\delta(\bar{x}, \bar{y}) = 1$ if $\bar{x} = \bar{y}$ and $\delta(\bar{x}, \bar{y}) = 0$ if $\bar{x} \neq \bar{y}$). We define the following Fourier transform:

$$\Im[V(\bar{x})] = f(\bar{k}) = \sum_{\bar{x} \in Z^n} V(\bar{x}) \mathrm{e}^{-i\bar{k}\bar{x}} \tag{2}$$

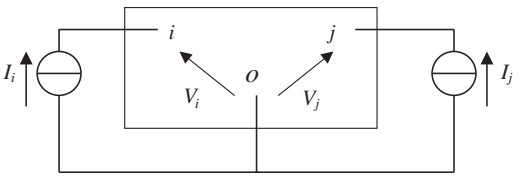

Figure 2. Scheme of the lattice network where two arbitrary nodes and the substrate node are considered and connected to the current generators described in the main text.

By using the following transformation rules:

$$\Im[V(\bar{x}+\bar{e}_h)] = f(\bar{k})\mathrm{e}^{\mathrm{i}k_h}; \quad \Im[V(\bar{x}-\bar{e}_h)] = f(\bar{k})\mathrm{e}^{\mathrm{i}k_h}; \quad \Im[\delta(\bar{x},\bar{x}^s)] = \mathrm{e}^{-\mathrm{i}\bar{k}\bar{x}^s} \tag{3}$$

we obtain the explicit solution in the transformed domain:

$$f(\bar{k}) = \frac{I_i\mathrm{e}^{-\mathrm{i}\bar{k}\bar{x}^i} + I_j\mathrm{e}^{-\mathrm{i}\bar{k}\bar{x}^j}}{\sum_{h=1}^{n} G_h[2-\mathrm{e}^{\mathrm{i}k_h}-\mathrm{e}^{-\mathrm{i}k_h}] + G_0} \tag{4}$$

The general expression for the inverse transform is given by

$$V(\bar{x}) = \Im^{-1}[f(\bar{k})] = \frac{1}{(2\pi)^n}\int_{-\pi}^{+\pi},\ldots,\int_{-\pi}^{+\pi} f(\bar{k})\mathrm{e}^{\mathrm{i}\bar{k}\bar{x}}\,\mathrm{d}\bar{k} \tag{5}$$

Therefore, substituting Equation (4) in Equation (5) we obtain, after some straightforward computations, the following integral expression for the electrical potential in an arbitrary node of the lattice network:

$$V(\bar{x}) = \frac{1}{(2\pi)^n}\int_{-\pi}^{+\pi},\ldots,\int_{-\pi}^{+\pi} \frac{I_i\mathrm{e}^{-\mathrm{i}\bar{k}\bar{x}^i} + I_j\mathrm{e}^{-\mathrm{i}\bar{k}\bar{x}^j}}{\sum_{h=1}^{n} G_h[2-\mathrm{e}^{\mathrm{i}k_h}-\mathrm{e}^{-\mathrm{i}k_h}] + G_0}\,\mathrm{e}^{\mathrm{i}\bar{k}\bar{x}}\,\mathrm{d}\bar{k} \tag{6}$$

This expression may be specialized to the two nodes of interest, by defining the potentials of the nodes $\bar{x}^i$ and $\bar{x}^j$, where the current generators have been connected. The potentials in these points are linearly related to the two current I_i and I_j by means of the impedance matrix $\tilde{Z}$ [14]:

$$\begin{cases} V_i = V(\bar{x}_i) \\ V_j = V(\bar{x}_j) \end{cases} \Rightarrow \begin{cases} V_i = \tilde{Z}_{ii}I_i + \tilde{Z}_{ij}I_j \\ V_j = \tilde{Z}_{ji}I_i + \tilde{Z}_{jj}I_j \end{cases} \tag{7}$$

By drawing a comparison between Equations (6) and (7) we may find out the explicit expressions for the impedance matrix elements:

$$\tilde{Z}_{ii} = \tilde{Z}_{jj} = \frac{1}{(2\pi)^n} \int_{-\pi}^{+\pi}, \ldots, \int_{-\pi}^{+\pi} \frac{1}{\sum_{h=1}^{n} G_h[2 - e^{ik_h} - e^{-ik_h}] + G_0} \, d\bar{k}$$

$$\tilde{Z}_{ij} = \frac{1}{(2\pi)^n} \int_{-\pi}^{+\pi}, \ldots, \int_{-\pi}^{+\pi} \frac{e^{i\bar{k}(\bar{x}^i - \bar{x}^j)}}{\sum_{h=1}^{n} G_h[2 - e^{ik_h} - e^{-ik_h}] + G_0} \, d\bar{k} \tag{8}$$

$$\tilde{Z}_{ji} = \frac{1}{(2\pi)^n} \int_{-\pi}^{+\pi}, \ldots, \int_{-\pi}^{+\pi} \frac{e^{i\bar{k}(\bar{x}^j - \bar{x}^i)}}{\sum_{h=1}^{n} G_h[2 - e^{ik_h} - e^{-ik_h}] + G_0} \, d\bar{k}$$

One can observe that the first formula in Equation (8) represents the value of the resistance between a generic node of the lattice and the node corresponding to the substrate. Furthermore, it can be noticed that the value of this resistance is invariant to any permutations of the values $G_1, G_2, \ldots, G_n$. It may be interesting to calculate the resistance between the nodes $\bar{x}^i$ and $\bar{x}^j$ if the node 0 of the substrate remains disconnected. In agreement with the conventions indicated in Figure 3, we may calculate this resistance in the following way:

$$Z_{ij} = \frac{V_i - V_j}{I_i} = \frac{\tilde{Z}_{ii}I_i + \tilde{Z}_{ij}I_j - (\tilde{Z}_{ji}I_i + \tilde{Z}_{jj}I_j)}{I_i} = \frac{\tilde{Z}_{ii}I_i - \tilde{Z}_{ij}I_i - \tilde{Z}_{ji}I_i + \tilde{Z}_{jj}I_i}{I_i} = 2\tilde{Z}_{ii} - \tilde{Z}_{ij} - \tilde{Z}_{ji} \tag{9}$$

By considering the relations given in Equation (8) the node–node resistance can be written as follows:

$$Z_{ij} = \frac{1}{(2\pi)^n} \int_{-\pi}^{+\pi}, \ldots, \int_{-\pi}^{+\pi} \frac{2 - e^{i\bar{k}(\bar{x}^i - \bar{x}^j)} - e^{-i\bar{k}(\bar{x}^i - \bar{x}^j)}}{\sum_{h=1}^{n} G_h[2 - e^{ik_h} - e^{-ik_h}] + G_0} \, d\bar{k} \tag{10}$$

For various applications that will be explained in the following sections we suppose that the nodes $\bar{x}^i$ and $\bar{x}^j$ are two adjacent nodes along the direction s of the n-dimensional lattice

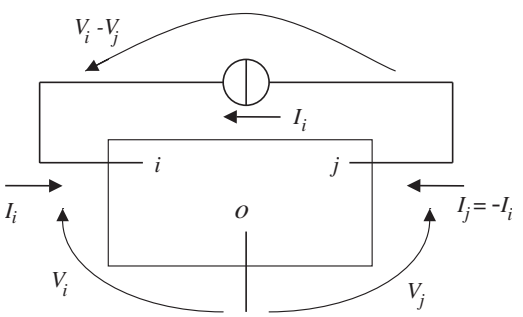

Figure 3. Particular configuration of the network introduced to define the node–node resistance. Here, the substrate node is an unconnected terminal.

$(1 \leqslant s \leqslant n)$. In this hypothesis we take into account, as main parameters of the lattice network, the node–substrate resistance and the node–node resistance between to adjacent nodes in direction s (see Figure 1):

$$R_{ns} = \frac{1}{\pi^n} \int_0^{+\pi}, \ldots, \int_0^{+\pi} \frac{\mathrm{d}\bar{k}}{2\sum_{h=1}^{n} G_h[1-\cos k_h] + G_0}$$
$$R_{nn}^{s} = \frac{1}{\pi^n} \int_0^{+\pi}, \ldots, \int_0^{+\pi} \frac{2[1-\cos k_s]\,\mathrm{d}\bar{k}}{2\sum_{h=1}^{n} G_h[1-\cos k_h] + G_0} \tag{11}$$

GENERAL EXPRESSIONS OF RESISTANCE INTEGRALS IN TERMS OF BESSEL FUNCTIONS

In this section we present a method to reduce the multiple integrals described in Equation (11) to single integrals performed on products of Bessel functions and exponential functions. This way of writing the resistance integrals is very useful for numerical application because of the fast convergence of such simple integrals. To begin we observe that the elementary property (Laplace transform),

$$\frac{1}{a} = \int_0^{\infty} \mathrm{e}^{-at}\,\mathrm{d}t, \quad a > 0 \tag{12}$$

may be used to convert the denominators in Equation (11) in exponential form. Now, the integrals over the variables $k_1, k_2, \ldots, k_n$ can be performed by using the following properties of the Bessel functions:

$$\pi I_0(\beta) = \int_0^{\pi} \mathrm{e}^{\beta\cos x}\,\mathrm{d}x; \quad \pi I_1(\beta) = \int_0^{\pi} \cos x \mathrm{e}^{\beta\cos x}\,\mathrm{d}x \tag{13}$$

After simple manipulations, the integrals describing the characteristic resistances are given by

$$R_{ns} = \frac{1}{2} \int_0^{\infty} \mathrm{e}^{-(G_0/2)t} \mathrm{e}^{-\sum_{h=1}^{n} G_h t} I_0(G_1 t) I_0(G_2 t), \ldots, I_0(G_n t)\,\mathrm{d}t \tag{14}$$

$$R_{nn}^{s} = \int_0^{\infty} \mathrm{e}^{-(G_0/2)t} \mathrm{e}^{-\sum_{h=1}^{n} G_h t} I_0(G_1 t) I_0(G_2 t), \ldots, [I_0(G_s t) - I_1(G_s t)], \ldots, I_0(G_n t)\,\mathrm{d}t \tag{15}$$

Equations (14) and (15) are completely equivalent to the starting points given in Equation (11). These forms of the resistance integrals may be very useful both for theoretical applications and for numerical procedures. Some considerations follow on the convergence of the R_{ns} integral given in Equation (14). To begin we recall the asymptotic behaviour of the zero-degree modified Bessel function: $I_0(z) \approx \mathrm{e}^z/\sqrt{2\pi z}$ when z tends to infinity [15]. If $G_0 \neq 0$, the convergence is assured by the presence of the exponential term with G_0. The problems appear when $G_0 = 0$. Apparently, from the absence of the conductances towards the substrate, we may deduce that the integral should be divergent because the substrate node is electrically disconnected from the other nodes of the lattice. However, the infinite dimension of the network could lead to some apparent paradoxes. Actually, when $G_0 = 0$, the

convergence of the integral depends on the dimensionality of the system. In fact, for n equal to 1 or 2 the R_{ns} integral is divergent, as expected, because one or two terms of the type $1/\sqrt{2\pi z}$ are not sufficient to permit the convergence. For $n \geqslant 3$ there are enough powers of $1/\sqrt{2\pi z}$ to give a converging integral. This fact that may appear as a paradox from the point of view of the electrical networks has an interesting explanation in the field of the random walks: it corresponds to the *recurrence*/*transience* transition [12, 13].

However, a particular case of lattice is given by isotropic networks; in such case $G = G_1 = G_2 = \cdots = G_n$ and Equations (14) and (15) can be written in a simplified form. A long but straightforward integration by parts allows us to write down a simplified result for the characteristic resistances of isotropic networks:

$$\begin{aligned} R_{ns} &= \frac{1}{2G}\int_0^\infty \mathrm{e}^{-(G_0/(2G))x}\mathrm{e}^{-nx} I_0^n(x)\,\mathrm{d}x \\ R_{nn} &= \frac{1}{nG}(1 - G_0 R_{ns}) \end{aligned} \tag{16}$$

In particular, this result means that there is a direct relation between the node–node and the node–substrate resistance in isotropic lattice networks. Moreover, with Equation (16) we may analyse the behaviour of isotropic networks without substrate ($G_0 \to 0$): in particular, a well-known problem in networks theory is the calculation of the impedance between adjacent nodes of infinite uniform n-dimensional resistive lattices. It is easy to observe that if $G_0 \to 0$, the product $G_0 R_{ns}$ tends to zero and therefore, form Equation (16) we obtain $R_{nn} = 1/(nG)$, a well-known result recently described, for example, in Reference [16].

GENERAL THEORY FOR RANDOM NETWORKS

We shall refer ourselves to n-dimensional lattice networks with substrate and we define an anisotropic distribution of conductance values by introducing different probability densities $\rho_k(G)$ for the conductances aligned along the different lattice directions k ($k = 1, \ldots, n$). Moreover, the conductances of the substrate are distributed following a given probability density $\rho_0(G)$. All the conductance values (each direction and substrate) are independently distributed according to the probability densities upon described. In the effective medium theory the average effects of the random conductances in such a disordered network will be represented by an anisotropic effective network in which all the conductances in k direction have the same value $\bar{G}_k$ and all the conductances in the substrate have the same value $\bar{G}_0$. These effective conductances will be self-consistently determined by the requirement that the fluctuating local potential in the random network should average to zero. Furthermore, for following purposes, we suppose to exactly know the functions $R_{nn}^k = R_{nn}^k(\bar{G}_0, \bar{G}_1, \ldots, \bar{G}_n)\ \forall k$ and $R_{ns} = R_{ns}(\bar{G}_0, \bar{G}_1, \ldots, \bar{G}_n)$ and to be able to compute them, almost numerically. They play a crucial role in determining the effective network equivalent to a random one. From now on, we suppose to know the effective network corresponding to a given random one, in order to understand the conceptual connection among them. In the effective network we change a single conductance $\bar{G}_k$, oriented along the direction k ($k = 1, \ldots, n$) or belonging to the substrate ($k = 0$), back to its true value G_k. This procedure can be applied indifferently to a resistor in the lattice or a resistor in the substrate. In Figure 4 one can find the graphical representation

of such a substitution, where the Thevenin equivalent circuit of the remaining part of the effective network is indicated. Here, G_k is a particular instance for the conductance value and $\bar{G}_k$ is the corresponding effective value. Moreover, V_{eq} and G_{eq} are the parameters of the Thevenin equivalent circuit. The electrical potentials on the circuits described in Figure 4 can be evaluated as follows:

$$\bar{V} = V_{\mathrm{eq}} \frac{G_{\mathrm{eq}}}{G_{\mathrm{eq}} + \bar{G}_k}; \quad V = V_{\mathrm{eq}} \frac{G_{\mathrm{eq}}}{G_{\mathrm{eq}} + G_k} \tag{17}$$

So, the potential fluctuations due to the random character of the network are given by

$$\Delta V = V - \bar{V} = V_{\mathrm{eq}} \frac{G_{\mathrm{eq}}}{G_{\mathrm{eq}} + G_k} - \bar{V} = \bar{V} \frac{G_{\mathrm{eq}} + \bar{G}_k}{G_{\mathrm{eq}}} \frac{G_{\mathrm{eq}}}{G_{\mathrm{eq}} + G_k} - \bar{V} = \bar{V} \frac{\bar{G}_k - G_k}{G_{\mathrm{eq}} + G_k} \tag{18}$$

Now, we may observe a relationship between the Thevenin conductance G_{eq} and the values R_{nn} and R_{ns}. In fact, G_{eq} is the conductance between the nodes **A** and **B** of Figure 4 where we have eliminated the conductance $\bar{G}_k$. Thus, in agreement with Figure 5, we may write down the relations:

$$\begin{aligned} k = 1, \ldots, n &\Rightarrow R_{nn}^k = \frac{1}{G_{\mathrm{eq}} + \bar{G}_k} \Rightarrow G_{\mathrm{eq}} = \frac{1}{R_{nn}^k} - \bar{G}_k \\ k = 0 &\Rightarrow R_{ns} = \frac{1}{G_{\mathrm{eq}} + \bar{G}_0} \Rightarrow G_{\mathrm{eq}} = \frac{1}{R_{ns}} - \bar{G}_0 \end{aligned} \tag{19}$$

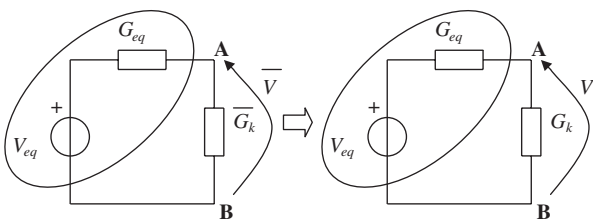

Figure 4. Thevenin equivalent circuit between two adjacent arbitrary nodes of the lattice before and after the substitution of the effective conductance $\bar{G}_k$ with a particular random instance G_k.

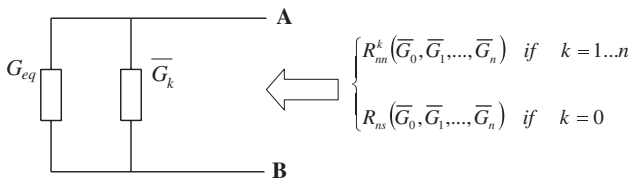

Figure 5. Scheme that defines the relation between the Thevenin conductance G_{eq} introduced in Figure 4 and the characteristic resistance values $R_{nn}^k = R_{nn}^k(\bar{G}_0, \bar{G}_1, \ldots, \bar{G}_n)\ \forall k$ and $R_{ns} = R_{ns}(\bar{G}_0, \bar{G}_1, \ldots, \bar{G}_n)$.

By substituting Equation (19) into Equation (18) and by imposing that the fluctuations of the potential average to zero, we have the homogenization integral equations:

$$\begin{aligned}
&\int_0^{+\infty} \rho_k(G) \frac{\bar{G}_k - G}{1/R_{nn}^k(\bar{G}_0, \bar{G}_1, \ldots, \bar{G}_n) - \bar{G}_k + G} \, \mathrm{d}G = 0 \quad \forall k = 1, \ldots, n \\
&\int_0^{+\infty} \rho_0(G) \frac{\bar{G}_0 - G}{1/R_{ns}(\bar{G}_0, \bar{G}_1, \ldots, \bar{G}_n) - \bar{G}_0 + G} \, \mathrm{d}G = 0
\end{aligned} \tag{20}$$

These relations represent a system of $n+1$ equations with $n+1$ unknowns $\bar{G}_0, \bar{G}_1, \ldots, \bar{G}_n$ that can be found when all the probability densities involved are given. The analysis of some random systems with the help of Equation (20) will be described in a following section, drawing a comparison between theoretical results and Monte-Carlo simulations. A first result is introduced here: the simpler network is the isotropic one without substrate. In this case only one value is unknown $\bar{G} = \bar{G}_1 = \cdots = \bar{G}_n$; the substrate is absent and so $\bar{G}_0$ is not of interest. Moreover, only one probability density describes the distribution of resistors in the lattice. We use the property for isotropic networks without substrate described in a previous section $R_{nn}(\bar{G}) = 1/(n\bar{G})$, and thus we obtain the simplest homogenizing result:

$$\int_0^{+\infty} \rho(G) \frac{1}{(n-1)\bar{G} + G} \, \mathrm{d}G = \frac{1}{n\bar{G}} \tag{21}$$

This is a well-known result in the field of the isotropic random networks [7, 9].

EXPLICIT EXPRESSIONS OF THE RESISTANCE INTEGRALS

In this section we report a series of exact results very useful to calculate the resistance R_{nn} and R_{ns} in closed form or in terms of elliptic integrals. For their definitions one can see classical books [15] or [17]. Some proofs of the results have been left in the appendices at the end of the text.

The one-dimensional case leads to very simple elementary functions; the resistance integrals (Equation (16) with $n = 1$) can be performed with the following property of Bessel functions [17]:

$$\int_0^{\infty} \mathrm{e}^{-\alpha x} I_0(\beta x) \, \mathrm{d}x = \frac{1}{\sqrt{\alpha^2 - \beta^2}}, \quad \alpha > \beta \tag{22}$$

So, the results follow:

$$R_{ns} = \frac{1}{\sqrt{G_0^2 + 4G_0 G}}; \quad R_{nn} = \frac{1}{G}\left[1 - \frac{G_0}{\sqrt{G_0^2 + 4G_0 G}}\right] \tag{23}$$

For two-dimensional structures the complete results are summarized below:

$$
\begin{aligned}
R_{ns}(G_0,G_1,G_2) &= \frac{r}{2\pi\sqrt{G_1G_2}}\,K(r)\\
R^1_{nn}(G_0,G_1,G_2) &= \frac{1}{G_1}\left\{1-\frac{2}{\pi}\left[K(r)E\left(\varepsilon,\sqrt{1-r^2}\right)-[K(r)-E(r)]F\left(\varepsilon,\sqrt{1-r^2}\right)\right]\right\}\\
R^2_{nn}(G_0,G_1,G_2) &= \frac{1}{G_2}\left\{1-\frac{2}{\pi}\left[K(r)E\left(\eta,\sqrt{1-r^2}\right)-[K(r)-E(r)]F\left(\eta,\sqrt{1-r^2}\right)\right]\right\}\\
\varepsilon &= \operatorname{arctg}\sqrt{\frac{4G_2+G_0}{4G_1}},\quad \eta=\operatorname{arctg}\sqrt{\frac{4G_1+G_0}{4G_2}},\quad r=4\sqrt{\frac{G_1G_2}{(4G_1+G_0)(4G_2+G_0)}}
\end{aligned}
\tag{24}
$$

The derivation of such results is given in Appendix A. When the two-dimensional lattice is isotropic $G=G_1=G_2$ the following simplified expressions hold on (see Appendix B):

$$
\begin{aligned}
R_{ns}(G_0,G,G) &= \frac{2}{\pi(4G+G_0)}\,K\left(\frac{4G}{4G+G_0}\right)\\
R_{nn}(G_0,G,G) &= \frac{1}{2G}\left\{1-\frac{2G_0}{\pi(4G+G_0)}\,K\left(\frac{4G}{4G+G_0}\right)\right\}
\end{aligned}
\tag{25}
$$

It can be observed that results in Equation (25) are in perfect agreement with Equation (16), which holds on for isotropic lattice networks. Furthermore, if the two-dimensional lattice network is without substrate ($G_0=0$), the exact formulas reduce to the following ones (see Appendix C):

$$
R^1_{nn}(0,G_1,G_2)=\frac{2}{\pi G_1}\operatorname{arctg}\sqrt{\frac{G_1}{G_2}};\quad R^2_{nn}(0,G_1,G_2)=\frac{2}{\pi G_2}\operatorname{arctg}\sqrt{\frac{G_2}{G_1}}
\tag{26}
$$

Finally, for three-dimensional structures the results can be written in terms of integrals of elliptic functions as follows:

$$
\begin{aligned}
R_{ns}(G_0,G_1,G_2,G_3) &= \frac{1}{2\pi^2}\sqrt{\frac{G_s}{G_1G_2G_3}}\int_0^{+\pi} r_s(x)K[r_s(x)]\,\mathrm{d}x\quad \forall s\\
R^s_{nn}(G_0,G_1,G_2,G_3) &= \frac{1}{\pi^2}\sqrt{\frac{G_s}{G_1G_2G_3}}\int_0^{+\pi}(1-\cos x)r_s(x)K[r_s(x)]\,\mathrm{d}x
\end{aligned}
\tag{27}
$$

where the function $r_1(x)$ is given by

$$
r_1(x)=4\sqrt{\frac{G_2G_3}{[4G_2+G_0+2G_1(1-\cos x)][4G_3+G_0+2G_1(1-\cos x)]}}
\tag{28}
$$

and the other ones $r_2(x)$ and $r_3(x)$ can be obtained from Equation (28) by permuting the indices in the symbols G_1, G_2 and G_3. This result for three-dimensional networks is a

generalization of that presented in Reference [18], with the presence of the substrate conductances G_0. The proof is very similar and it is omitted here for sake of brevity.

SOME EXAMPLES OF SIMULATIONS FOR DISORDERED SYSTEMS

To begin we take into consideration one-dimensional systems that correspond to random ladder networks. A first numerical experiment can be conducted by taking into account the following statistical rule for assigning the conductances in the network: we consider the same distribution function for the horizontal conductances (G) and the vertical ones (G_0). Each conductance is placed into the ladder network assuming the value x with probability c and value 1 with probability $1-c$. It means that c is the mean fraction of conductances with value x. In our simulations we will use the value $c=\frac{1}{2}$. General results given in Equation (20) and expressions for R_{nn} and R_{ns} given in Equation (23) lead immediately to the following system for the equivalent conductance values:

$$\begin{aligned}\frac{1}{\bar{G}}\left[1-\frac{\bar{G}_0}{\sqrt{\bar{G}_0^2+4\bar{G}_0\bar{G}}}\right]&=\frac{(\bar{G}-1)+c(1-x)}{(\bar{G}-1)(\bar{G}-x)}\\ \frac{1}{\sqrt{\bar{G}_0^2+4\bar{G}_0\bar{G}}}&=\frac{(\bar{G}_0-1)+c(1-x)}{(\bar{G}_0-1)(\bar{G}_0-x)}\end{aligned} \tag{29}$$

In general, the system can be reduced to a fifth-degree algebraic equation; in the case with $c=\frac{1}{2}$, the solution is simpler and the problem can be reduced to a third-degree algebraic equation. In fact, after some straightforward calculations is evident that if $c=\frac{1}{2}$, the system is equivalent to the following one:

$$\begin{aligned}&(4x+4)\bar{G}_0^3+(2x-3-3x^2)\bar{G}_0^2-(8x^2+8x)\bar{G}_0+(4x^3+4x^2+4x)=0\\ &\bar{G}=\frac{(3\bar{G}_0+3\bar{G}_0x-2x-4\bar{G}_0^2)(\bar{G}_0+\bar{G}_0x-2x)}{4\bar{G}_0(2\bar{G}_0-1-x)}\end{aligned} \tag{30}$$

Once the first equation is solved for $\bar{G}_0$ the second relationship may be directly applied to obtain the average value for the conductance $\bar{G}$. As macroscopic output of the system we take into consideration the input resistance R_{in} of the ladder network as defined in Figure 7 (internal schematic). So, we may draw a comparison between the average input resistance obtained with Monte-Carlo simulations on random networks and the equivalent counterpart written in terms of $\bar{G}_0$ and $\bar{G}$, given by $R_{\text{in}}=[1/(2\bar{G})][1+(1/\bar{G}_0)\sqrt{\bar{G}_0^2+4\bar{G}_0\bar{G}}]$. The results have been shown in Figures 6 and 7: more precisely, in Figure 6 one can find the behaviour of $\bar{G}_0$ and $\bar{G}$ obtained solving system given in Equation (30). In Figure 7 a comparison between theoretical values of R_{in} and obtained with the Monte-Carlo method is drawn, showing a good agreement between theory and simulations.

Another numerical experiment has been performed fixing the value of all the vertical conductances (G_0) and randomly assigning the horizontal ones. Each horizontal conductance is placed into the ladder network assuming the value x with probability $\frac{1}{2}$ and value 1 with

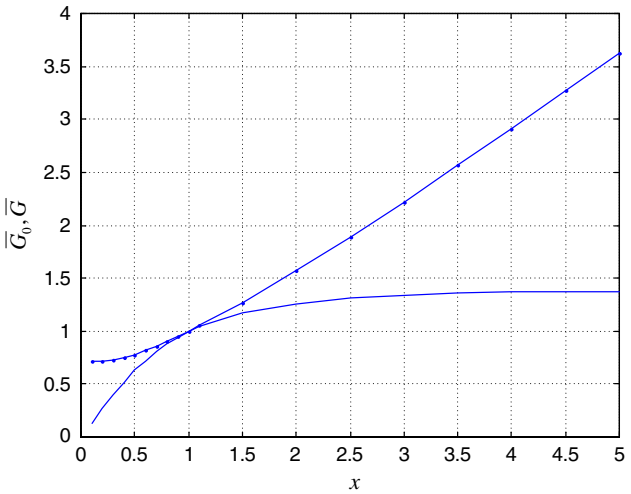

Figure 6. Behaviour of the effective conductances $\bar{G}_0$ (continuous dotted line) and $\bar{G}$ (continuous line) *versus* x obtained solving the system given in Equation (30). We considered a random ladder network with random horizontal and vertical conductances. Each conductance is placed into the ladder network assuming the value x with probability $\frac{1}{2}$ and value 1 with probability $\frac{1}{2}$.

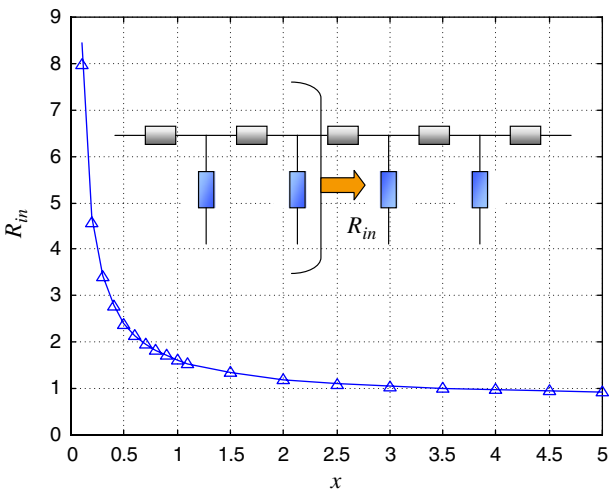

Figure 7. Monte-Carlo simulations (triangles) *versus* theoretical results (continuous line) for a ladder network with random horizontal and vertical conductances. The input resistance R_{in} of the ladder network (see schematic) is represented *versus* the values of x.

probability $\frac{1}{2}$. In our simulations we will use the value $G_0 = \frac{1}{10}$. Only one equation describes the behaviour of the effective horizontal conductance:

$$(-4G_0)\bar{G}^3 + (8G_0x + 8G_0 + 8x + 4x^2 + 4)\bar{G}^2 - (3G_0x^2 + 14G_0x - 3G_0 - 16x^2 - 16x)\bar{G} + (16x^2 + 4G_0x^2 + 4G_0x) = 0 \qquad (31)$$

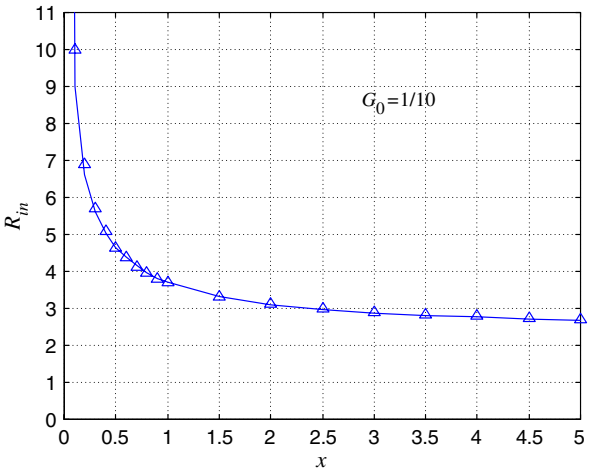

Figure 8. Monte-Carlo simulations (triangles) *versus* theoretical results (continuous line) for a ladder network with random horizontal conductances. Each horizontal conductance is placed into the ladder network assuming the value x with probability $\frac{1}{2}$ and value 1 with probability $\frac{1}{2}$. In simulations we used the fixed value $G_0 = \frac{1}{10}$. The input resistance R_{in} of the ladder network is represented *versus* the values of x.

Once again, we have drawn a comparison between theory and Monte-Carlo simulations. The comparison for the input resistance R_{in}, defined as before, is shown in Figure 8, yielding a further confirmation of the theory.

The final one-dimensional numerical experiment has been performed fixing the value of all the horizontal conductances (G) and randomly assigning the vertical ones. Each vertical conductance is placed into the ladder network assuming the value x with probability $\frac{1}{2}$ and value 1 with probability $\frac{1}{2}$. In our simulations we will use the value $G = \frac{1}{10}$. The equation for the effective vertical conductances is given by

$$(4x + 4 + 16G)\bar{G}_0^3 - (3 + 14x + 3x^2 + 16G + 16xG)\bar{G}_0^2 + (8x + 8x^2 + 4G + 8xG + 4x^2G)\bar{G}_0 - 4x^2 = 0 \tag{32}$$

Finally, we have drawn a comparison between input resistance values R_{in} obtained by theory and by Monte-Carlo simulations. One can find the results in Figure 9, where a good agreement is evident.

From now on we consider some examples of disordered two-dimensional lattices. A first series of simulations deal with the problem of the anisotropic percolation in two-dimensional systems. We consider a two-dimensional lattice grid without substrate with an anisotropic statistical distribution of the conductance values in the network. On the rows of the network (axis x) we assign conductances of value G_{0x} with probability $1 - c_x$ and value zero with probability c_x. On the columns (axis y) we assign conductances with value G_{0y} with probability $1 - c_y$ and value zero with probability c_y. Equations (20) and (26) may be applied to the lattice grid in order to find out the equivalent conductances that must be assigned to

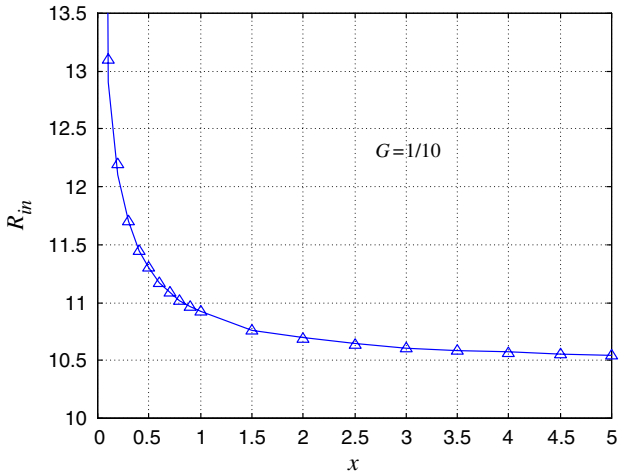

Figure 9. Monte-Carlo simulations (triangles) *versus* theoretical results (continuous line) for a ladder network with random vertical conductances. Each vertical conductance is placed into the ladder network assuming the fixed value x with probability $\frac{1}{2}$ and value 1 with probability $\frac{1}{2}$. In our simulations we used the value $G=\frac{1}{10}$. The input resistance R_{in} of the ladder network is represented *versus* the values of x.

the effective network. The substrate is absent, and thus the relative equation is not considered here. After some straightforward manipulations, the system for the effective conductances $\bar{G}_x$ and $\bar{G}_y$ may be recast in the simple form:

$$\begin{aligned} \operatorname{arctg}\sqrt{\frac{\bar{G}_x}{\bar{G}_y}} &= \frac{\pi}{2}\frac{\bar{G}_x-(1-c_x)G_{0x}}{\bar{G}_x-G_{0x}} \\ \bar{G}_y &= G_{0y}\frac{\bar{G}_x(1-c_y)+G_{0x}(c_x+c_y-1)}{\bar{G}_x-(1-c_x)G_{0x}} \end{aligned} \tag{33}$$

To perform the Monte-Carlo simulations we use the values $G_{0x}=G_{0y}=1$ and we analyse the behaviour of the system *versus* the volume fractions c_x and c_y. In Figure 10 the solutions of the system given in Equation (33) are shown (always with $G_{0x}=G_{0y}=1$) in terms of c_x and c_y. It is interesting to note that the percolation threshold for this system is given by $c_x+c_y=1$. In Figure 11 the comparison between theoretical results (continuous lines) and simulations (triangles) is shown. Here, the effective conductances in both spatial directions are plotted *versus* the first concentration c_x. Different curves represent different values of the other concentration c_y. In both plots from the top to the bottom the curves correspond to the following values of c_y: 0.027–0.185–0.342–0.5–0.657–0.815. A good agreement between theory and simulations has been obtained.

Now, we devote our attention to two cases of two-dimensional networks with substrate that are very important for practical applications. The first case deals with a two dimensional isotropic grid where the conductances of the lattices are randomly placed and the conductances of the substrate are all fixed at a given value. Each conductance of the grid is placed into

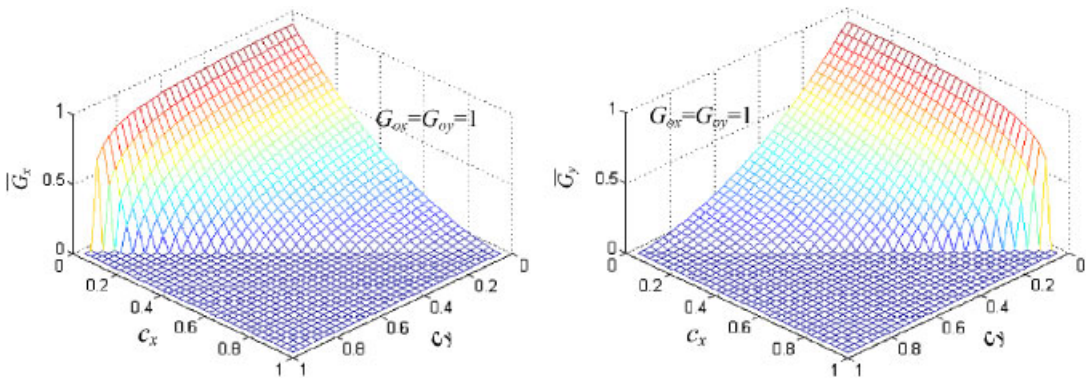

Figure 10. Theoretical results (solutions of Equation (33) for the effective conductances $\bar{G}_x$ and $\bar{G}_y$) for two-dimensional anisotropic percolation. On the rows of the network we assigned conductances of value 1 with probability $1 - c_x$ and value zero with probability c_x. On the columns we assigned conductances with value 1 with probability $1 - c_y$ and value zero with probability c_y. One can note that the percolation threshold for this system is given by $c_x + c_y = 1$.

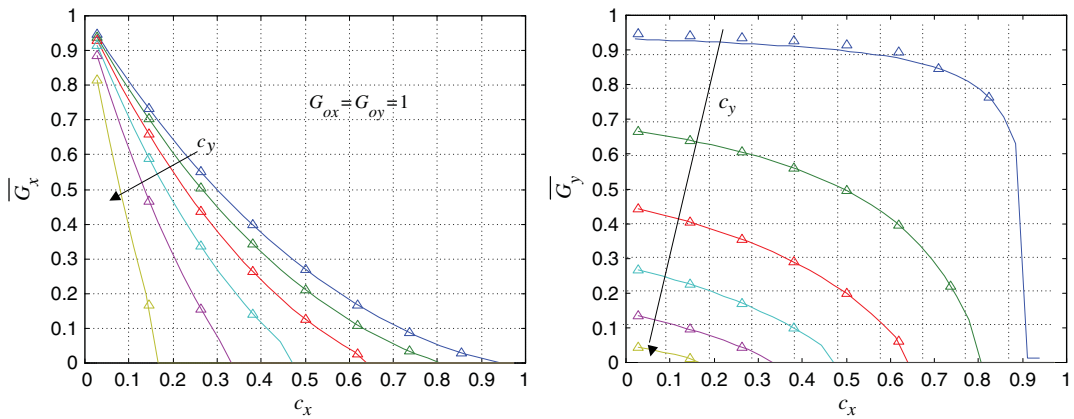

Figure 11. Comparison between theoretical results (continuous lines) and simulations (triangles) for anisotropic percolation. The effective conductances in both spatial directions are plotted *versus* the first concentration c_x. Different curves represent different values of the other concentration c_y. In both plots from the top to the down the curves correspond to the following values of c_y: 0.027−0.185−0.342−0.5−0.657−0.815.

the network assuming the value G_1 with probability $\frac{1}{2}$ and the value G_2 with probability $\frac{1}{2}$. All the substrate conductances are fixed to the value G_0. The combination of Equations (20) and (25) leads, after some simple manipulations, to the following equation for the unknown effective conductance $\bar{G}$ of the grid:

$$K\left(\frac{4\bar{G}}{4\bar{G}+G_0}\right)=\frac{\pi}{2}\,\frac{(G_1G_2-\bar{G}^2)(4\bar{G}+G_0)}{G_0(\bar{G}-G_1)(\bar{G}-G_2)} \tag{34}$$

This result is very interesting because the parameter G_0 modulates the kind of mean value $\bar{G}$ between G_1 and G_2; the extreme cases are the following: if G_0 is zero the value, $\bar{G}$ corresponds to the *geometric mean* between G_1 and G_2, $\bar{G} = \sqrt{G_1 G_2}$, and if G_0 tends to infinity, the value $\bar{G}$ corresponds to the *arithmetic mean* between G_1 and G_2, $\bar{G} = (G_1 + G_2)/2$. In other words, Equation (34) defines a family of mean values that depend on the value of G_0. Simulations with $G_1 = 1$ and $G_2 = 4$ have been performed. A comparison between results obtained from Equation (34) and Monte-Carlo simulations is shown in Figure 12 where theoretical values of R_{ns} and numerical ones are shown *versus* values of G_0 between 1 and 100. A remarkably good fitting is evident.

The second case of two-dimensional lattices on substrate deals with a grid with fixed conductances in both directions and random conductances in the substrate. Each conductance of the substrate is placed into the network assuming the value G_1 with probability $\frac{1}{2}$ and the value G_2 with probability $\frac{1}{2}$. All the lattice conductances are fixed to the value G. As before, the combination of Equations (20) and (25) leads, after some simple manipulations, to the following equation for the unknown effective conductance $\bar{G}_0$ of the grid:

$$K\left(\frac{4G}{4G+\bar{G}_0}\right) = \frac{\pi}{2}\frac{\left(\bar{G}_0 - \dfrac{G_1+G_2}{2}\right)(4G+\bar{G}_0)}{(\bar{G}_0 - G_1)(\bar{G}_0 - G_2)} \tag{35}$$

This result is similar to that given in Equation (34) because the parameter G modulates the kind of mean value $\bar{G}_0$ between G_1 and G_2 also in this case; the extreme cases are the

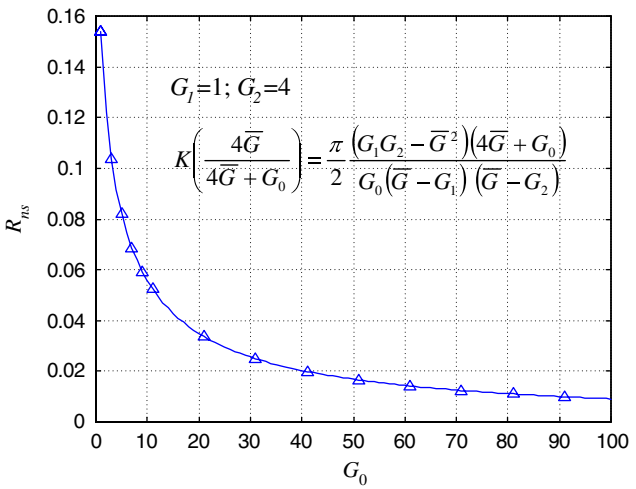

Figure 12. Theoretical (continuous line) and Monte-Carlo results (triangles) for a two-dimensional grid where the conductances of the lattices are randomly placed and the conductances of the substrate are all fixed at a given value. Each conductance of the grid is placed into the network assuming value 1 with probability $\frac{1}{2}$ and the value 4 with probability $\frac{1}{2}$. All the substrate conductances are fixed to the value G_0. The node–substrate resistance R_{ns} of the network is represented *versus* the values of G_0.

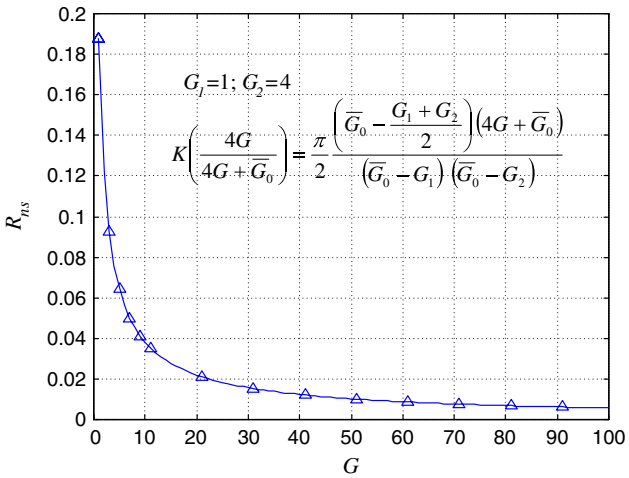

Figure 13. Theoretical (continuous line) and Monte-Carlo results (triangles) for a two dimensional grid where the conductances of the lattices are fixed and the conductances of the substrate are randomly assigned. Each conductance toward the substrate is placed into the network assuming value 1 with probability $\frac{1}{2}$ and value 4 with probability $\frac{1}{2}$. All the grid conductances are fixed to the value G. The node–substrate resistance R_{ns} of the network is represented *versus* the values of G.

following: if G is zero, the value $\bar{G}_0$ corresponds to the *harmonic mean* between G_1 and G_2, $\bar{G}_0 = 2G_1G_2/(G_1 + G_2)$, and if G tends to infinity, the value $\bar{G}_0$ corresponds to the *arithmetic mean* between G_1 and G_2, $\bar{G}_0 = (G_1 + G_2)/2$. Finally, simulations with $G_1 = 1$ and $G_2 = 4$ have been performed. A comparison between results obtained from Equation (35) and Monte-Carlo simulations is shown in Figure 13 where theoretical values of R_{ns} and numerical ones are shown *versus* values of G between 1 and 100.

CONCLUSIONS

We have developed an effective medium theory for a general disordered lattice network of resistors. The theoretical predictions have been verified by means of a series of Monte-Carlo analyses with remarkably good agreement. The general approach takes into account a new topology of networks that describe the effects of a substrate coupled to the resistors lattice. This is an important point in many applications such as films of heterogeneous materials deposited on substrates with electrical interaction or random ladder networks. An interesting field of application of the present theory is given by the analysis of the anisotropic percolation. Some two-dimensional examples have been described in the main text. Moreover, as additional results we have reported many exact relations describing the electrical behaviour of homogeneous but anisotropic lattice systems of various dimensionality. These results are based on the Green's lattice function expressed in terms of elliptic integrals and may be useful for different applications.

APPENDIX A: TWO-DIMENSIONAL ANISOTROPIC NETWORKS WITH SUBSTRATE

For two-dimensional anisotropic networks the equivalent resistance integrals described in the main text can be explicitly written as follows:

$$\begin{aligned} R_{ns}(G_0,G_1,G_2) &= \frac{1}{\pi^2}\int_0^{+\pi}\int_0^{+\pi}\frac{\mathrm{d}k_1\,\mathrm{d}k_2}{2G_1[1-\cos k_1]+2G_2[1-\cos k_2]+G_0} \\ R^s_{nn}(G_0,G_1,G_2) &= \frac{1}{\pi^2}\int_0^{+\pi}\int_0^{+\pi}\frac{2[1-\cos k_s]\,\mathrm{d}k_1\,\mathrm{d}k_2}{2G_1[1-\cos k_1]+2G_2[1-\cos k_2]+G_0} \end{aligned} \tag{A1}$$

We start with the evaluation of the first integral R_{ns} performing the integration over the variable k_1; this integral can be evaluated with the elementary formula [17]:

$$\int_0^{+\pi}\frac{\mathrm{d}x}{a+b\,\cos x}=\frac{\pi}{\sqrt{a^2-b^2}},\quad a>|b| \tag{A2}$$

Therefore, we obtain the following result:

$$R_{ns}(G_0,G_1,G_2)=\frac{1}{\pi}\int_0^{+\pi}\frac{\mathrm{d}k_2}{\sqrt{[2G_2(1-\cos k_2)+G_0][2G_2(1-\cos k_2)+G_0+4G_1]}} \tag{A3}$$

Now, the remaining integration may be approached by means of the following substitution, which defines a new variable $y=1-\cos k_2 \Rightarrow \mathrm{d}k_2=\mathrm{d}y/\sqrt{y(2-y)}$. This operation leads to the following expression:

$$R_{ns}(G_0,G_1,G_2)=\frac{1}{2\pi G_2}\int_0^2\frac{\mathrm{d}y}{\sqrt{2-y}\sqrt{y}\sqrt{y+\dfrac{G_0}{2G_2}}\sqrt{y+\dfrac{G_0+4G_1}{2G_2}}} \tag{A4}$$

The last integral is of the form $\int 1/\sqrt{P(y)}\,\mathrm{d}y$ where $P(y)$ is a fourth-degree polynomial; therefore, it can be reduced to elliptic integrals, as follows (see Reference [17, p. 242]):

$$\begin{aligned} &\int_u^a\frac{\mathrm{d}x}{\sqrt{(a-x)(x-b)(x-c)(x-d)}}=\frac{2}{\sqrt{(a-c)(b-d)}}F(\mu,r) \\ &a>u\geqslant b>c>d,\quad \mu=\operatorname{arcsen}\sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}},\quad r=\sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}} \end{aligned} \tag{A5}$$

In order to apply the general solution given in Equation (A5) we define the following parameters drawing a comparison with Equation (A4): $a=2,\ u=b=0,\ c=-G_0/(2G_2)$, $d=-(G_0+4G_1)/(2G_2)$. Consequently, we obtain the following values for the auxiliary parameters μ and r: $\mu=\pi/2,\ r=4\sqrt{G_1G_2}/\sqrt{(4G_1+G_0)(4G_2+G_0)}$. Summing up, using the relation $K(k)=F(\pi/2,k)$, we obtain the final result for the node–substrate resistance in two-dimensional anisotropic networks with substrate as given in Equation (24) of the text.

From now on, we take into consideration the second integral describing R_{nn}; we refer to the node–node resistance along the first spatial direction and we will obtain the other one by means of cyclic permutation of the indices. As before, the first integration in the second integral of Equation (A1) can be performed by using the property given in Equation (A2), obtaining:

$$R^1_{nn}(G_0,G_1,G_2)=\frac{1}{\pi}\int_0^{+\pi}\frac{2(1-\cos k_1)\,\mathrm{d}k_1}{\sqrt{[2G_1(1-\cos k_1)+G_0][2G_1(1-\cos k_1)+G_0+4G_2]}} \tag{A6}$$

At this point we can proceed by evaluating the integration over k_1 by means of the following substitution: $y=1-\cos k_1\Rightarrow \mathrm{d}k_1=\mathrm{d}y/\sqrt{y(2-y)}$. The integral is transformed in the following one:

$$R^1_{nn}=\frac{1}{\pi G_1}\int_0^2\frac{\sqrt{y}\,\mathrm{d}y}{\sqrt{2-y}\sqrt{y+\dfrac{G_0}{2G_1}}\sqrt{y+\dfrac{G_0+4G_2}{2G_1}}} \tag{A7}$$

Once again, this is an integration that can be reduced to elliptic integrals; we may use the following general rule (see Reference [17, p. 265]):

$$\int_b^u\sqrt{\frac{(x-b)}{(a-x)(x-c)(x-d)}}\,\mathrm{d}x=\frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\lambda,\frac{a-b}{a-c},r\right)F(\lambda,r)\right]$$
$$a\geqslant u>b>c>d,\quad \lambda=\operatorname{arcsen}\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}},\quad r=\sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}} \tag{A8}$$

In order to apply Equation (A8) we define the following parameters (see terms in Equation (A7) for comparison): $a=u=2,\ b=0,\ c=-G_0/(2G_1),\ d=-(G_0+4G_2)/(2G_1)$. The corresponding values for λ and r are given by: $\lambda=\pi/2,\ r=4\sqrt{G_1G_2}/\sqrt{(4G_1+G_0)(4G_2+G_0)}$. Thus a first result is given by the following expression:

$$R^1_{nn}(G_0,G_1,G_2)=\frac{2G_0}{\pi G_1\sqrt{(G_0+4G_1)(G_0+4G_2)}}\left[\Pi\left(\frac{\pi}{2},\frac{4G_1}{4G_1+G_0},r\right)-K(r)\right] \tag{A9}$$

In this result a complete elliptic integral of the third kind appears; it can be simplified by means of the following relationship that holds on when $r^2<n<1$ (this condition is known as

circular case, see Reference [15, p. 599]):

$$\Pi\left(\frac{\pi}{2},n,r\right)-K(r)=\frac{1}{2}\,\pi\sqrt{\frac{n}{(1-n)(n-r^2)}}\left\{1-\frac{2}{\pi}\left[K(r)E\left(\varepsilon,\sqrt{1-r^2}\right)\right.\right.$$

$$\left.\left.-[K(r)-E(r)]F\left(\varepsilon,\sqrt{1-r^2}\right)\right]\right\} \tag{A10}$$

where $\varepsilon=\text{arcsen}(\sqrt{1-n}/\sqrt{1-r^2})=\text{arctg}(\sqrt{1-n}/\sqrt{n-r^2})$ is the argument of the incomplete elliptic integrals E and F. In our case we apply Equation (A10) with $n=4G_1/(4G_1+G_0)$; finally, we arrive at the results shown in Equation (24).

APPENDIX B: FIRST LIMITING CASE FOR TWO-DIMENSIONAL LATTICE: ISOTROPIC NETWORK WITH $G_1=G_2=G$

The general expressions given in Equation (24) may be strongly simplified when we are dealing with an isotropic network characterized by $G_1=G_2=G$. Under this hypothesis, we observe the following simplifications of the involved quantities:

$$G_1=G_2=G\Rightarrow\varepsilon=\eta=\text{arctg}\sqrt{\frac{4G+G_0}{4G}},\quad r=\frac{4G}{4G+G_0}\Rightarrow\varepsilon=\eta=\text{arctg}\sqrt{\frac{1}{r}} \tag{B1}$$

When the argument and the modulus of incomplete elliptic integrals are related as indicated in (B1) some useful expressions [15] help us to handle the problem:

$$2F\left(\text{arctg}\sqrt{\frac{1}{r}},\sqrt{1-r^2}\right)=K\left(\sqrt{1-r^2}\right);\ 2E\left(\text{arctg}\sqrt{\frac{1}{r}},\sqrt{1-r^2}\right)=E\left(\sqrt{1-r^2}\right)+1-r \tag{B2}$$

In order to simplify the notations we define the complementary modulus as follows: $\sqrt{1-r^2}=r'$. Relations (B2) allow us to simplify the expression of the node–node resistance:

$$R_{nn}=R^1_{nn}=R^2_{nn}=\frac{1}{G}\left\{1-\frac{2}{\pi}\left[K(r)E\left(\text{arctg}\sqrt{\frac{1}{r}},r'\right)-[K(r)-E(r)]F\left(\text{arctg}\sqrt{\frac{1}{r}},r'\right)\right]\right\}$$

$$=\frac{1}{G}\left\{1-\frac{2}{\pi}\left[K(r)\frac{E(r')+1-r}{2}-[K(r)-E(r)]\frac{K(r')}{2}\right]\right\}$$

$$=\frac{1}{2G}\left\{1-\frac{2}{\pi}(1-r)K(r)\right\} \tag{B3}$$

where the Legendre relation $E(r)K(r') + K(r)E(r') - K(r)K(r') = \pi/2$ for complete elliptic integrals has been used. So, final formulas for isotropic network have been verified as appear in Equation (25).

APPENDIX C: SECOND LIMITING CASE FOR TWO-DIMENSIONAL LATTICE: ANISOTROPIC NETWORK WITHOUT SUBSTRATE

We start again with the general result, Equation (24), and we consider the simplified problem with absence of substrate, which means $G_0 = 0$. In this computation we maintain the anisotropy of the network, i.e. we can consider $G_1 \neq G_2$. To perform the computation we refer to the expression for R^1_{nn}; the other formula giving R^2_{nn} will be obtained with cyclic permutation of the indices.

In the limit of $G_0 \to 0$ we have these following values of the various parameters: $\varepsilon \to \operatorname{arctg}\sqrt{G_2/G_1}$, $r \to 1$ and $r' = \sqrt{1 - r^2} \to 0$. When the modulus of the elliptic integrals tends to zero the following relations hold in [15]: $F(\varepsilon, 0) = \varepsilon$, $E(\varepsilon, 0) = \varepsilon$. We use such properties in order to simplify Equation (34)

$$
\begin{aligned}
R^1_{nn}(0, G_1, G_2) &= \frac{1}{G_1}\left\{1 - \frac{2}{\pi}[K(r)\varepsilon - [K(r) - E(r)]\varepsilon]\right\} \\
&= \frac{1}{G_1}\left\{1 - \frac{2}{\pi}E(r)\varepsilon\right\} = \frac{1}{G_1}\left\{1 - \frac{2}{\pi}\varepsilon\right\} = \frac{1}{G_1}\left\{1 - \frac{2}{\pi}\operatorname{arctg}\sqrt{\frac{G_2}{G_1}}\right\} \\
&= \frac{2}{\pi G_1}\operatorname{arctg}\sqrt{\frac{G_1}{G_2}} \qquad \text{(C1)}
\end{aligned}
$$

We have also recalled the particular value $E(1) = 1$. As expected, the value of R_{ns} diverges to infinity. Therefore, the final results for anisotropic two-dimensional networks without substrate are given by Equation (26).

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