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Effects of the orientational distribution of cracks in isotropic solids

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Abstract

The paper deals with the elastic characterisation of microcracked solids: we analyse dispersions of cracks with arbitrary non-random orientational distributions. Particular cases of angular distributions are given by cracks all oriented in a given direction or cracks uniformly random oriented in the space. A unified theory covers all the orientational distributions between the random and the parallel ones. The micromechanical averaging inside the composite material is carried out by means of explicit results which allows us to obtain closed-form expressions for the macroscopic or equivalent elastic moduli of the overall material. The analysis has been performed in two-dimensional (2D) elasticity (plane stress and plane strain) with slit like cracks and in three-dimensional (3D) elasticity with planar circular cracks. The elastic behaviour of the microcracked solid depends upon the density of cracks and upon their orientational distribution. In particular, this study allows us to state that in two-dimensions the elastic behaviour of such a microcracked material is completely defined by one order parameter, which depends on the given angular distribution while the elastic characterisation in three-dimensions depends on two order parameters. The particular cases of isotropic orientations of cracks (both in 2D and in 3D) have been generalised to higher values of the cracks density by means of the method of the iterated homogenisation, which leads to some differential equations. Their solutions show that the equivalent elastic moduli depend exponentially on the cracks density.

Keywords: Cracks distribution; Elastic homogenisation; Fracture mechanics; Order parameters

1. Introduction

In this paper we study the behaviour of an elastic medium containing cracks in the framework of micromechanics. The presence of cracks in a homogeneous elastic matrix will influence the mechanical properties of the medium. Thus, by modeling the crack distribution inside a material we may study the effects of the cracks on the mechanical properties, such as the elastic moduli (the stiffness tensor or the compliance tensor).

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Nomenclature

x, y, z	orthonormal coordinates
a_x, a_y, a_z	semi-axes of the ellipsoidal void
e	aspect ratio of the ellipsoids
a	half-length of slit-cracks in 2D elasticity and radius of circular cracks in 3D elasticity
$\mathbf{T}$, T_{ij}	stress tensor and its entries
$\mathbf{E}$, E_{ij}	strain tensor and its entries
$\mathbf{L}^1$, L^1_{ijkh}	stiffness tensor and its entries
$\widehat{\mathbf{T}}$	stress tensor in Voigt notation
$\widehat{\mathbf{E}}$	strain tensor in Voigt notation
$\widehat{\mathbf{L}}^1$	stiffness tensor in Voigt notation
E, ν	Young modulus and Poisson's ratio of the isotropic matrix
k, μ	bulk modulus and shear modulus of the isotropic matrix
$\widehat{\mathbf{E}}_i$	internal strain tensor in Voigt notation
$\widehat{\mathbf{E}}_0$	external strain tensor in Voigt notation
$\widehat{\mathbf{S}}$, s_{ijkh}	Eshelby's tensor in Voigt notation and its entries
$\widehat{\mathbf{C}}$, c_{ijkh}	averaged Wu's tensor and its entries
$\widehat{\mathbf{G}}$, g_{ijkh}	limiting value of the Wu's tensor and relative entries
$\widehat{\mathbf{L}}_{\mathrm{eq}}$, L_{ijkh}	effective stiffness tensor and its entries
$\mathbf{I}$	identity tensor
$\mathbf{R}$	rotation matrix
$\widehat{\mathbf{M}}$	rotation matrix in Voigt notation
θ, ψ, φ	rotation angles
$f(\theta)$	density probability for the angle θ
$\delta(\theta)$	Dirac delta function
P	order parameter for 2D elasticity
S, T	order parameters for 3D elasticity
$P_n(\cos\theta)$	Legendre polynomials of order n
n, l, k, m and p	Hill parameters for transversely isotropic media
N	number of cracks
ΔN	increment of the number of cracks
c	volume fraction
A	area in 2D elasticity
V	volume in 3D elasticity
α	cracks density (both in 2D and in 3D elasticity)
k_{eq}, μ_{eq}	effective bulk modulus and shear modulus
E_{eq}, ν_{eq}	effective Young modulus and Poisson's ratio

For many practical applications one may then attempt to invert measurements of such properties for deriving crack parameters such as dimensions and orientations. Clearly, then, advancements in the modeling of the mechanical properties resulting from a distribution of cracks will aid the accuracy of inversion techniques.

The characterisation of cracked materials belongs to the vast field of homogenisation in composite materials [1,2]. For example, dealing with elastic characterisation of dispersions [3], many works have been developed: the most famous and studied elastic mixture theory regards a composite material formed by spherical inclusions embedded in a solid matrix. An exact result exists for such a material composed by a very dilute concentration of spherical inclusions (with bulk modulus k_2 and shear modulus μ_2) dispersed in a solid matrix (with moduli k_1 and μ_1). This result is attributed to different authors [4]. Moreover, many other works have been devoted to the analysis of the effects of ellipsoidal inclusions in a given matrix [5]. To adapt the dilute formulas to the case of any finite volume fraction a great number of proposals have been made and they appear in technical literature. The most useful approaches are the iterated homogenisation [6] and the differential effective medium theory [7,8].

In the present work, in the conceptual framework of the homogenisation, in order to model the flat shape of a crack, we adopt an ellipsoidal void with an axis with infinitesimal length. Treating the crack as a vacuous oblate ellipsoid of eccentricity approaching zero is very convenient. The idea is that one can derive the needed formulas for a microcracked solid, passing (with a due care) to the limits in the general formulas, concerning ellipsoidal inclusions. This approach is not new in principle and it has been used in various homogenisation theories for microcracked media [9–11]. In such relevant works the orientational distribution of cracks is given by one of the two most adopted distributions: cracks aligned with a given direction or cracks uniformly oriented in the space. In some other recent works [12,13] it has been verified analytically and experimentally that irregularities on the angular distribution of cracks in a given medium can produce pronounced effects on the effective properties. In such works, in order to mimic the microstructure of plasma-sprayed coatings, the authors consider families of penny shaped cracks having an orientational scatter about some preferential orientations. To do this, they introduce a given probability density function for the statistical distribution of the angles defining the orientation of each crack. Such a density function shows a scatter parameter characterising the state of order or disorder of the angular distribution of the cracks. So doing, they are able to take into account different orientational distributions between the extreme cases of the fully random and of the ideally parallel cracks. The aim of the present paper is to analyse a microcracked solid with a completely arbitrary angular distribution of cracks. So, a particular attention is devoted to the analysis of the effects of the orientational distribution of the cracks inside the damaged material. The limiting cases of the present theory are represented by all the cracks aligned with a given direction (order) and all the cracks randomly oriented (disorder). We take into account all the intermediate configurations between order and disorder with the aim to characterise a material with cracks partially aligned. Two different cases have been taken into consideration: the two-dimensional distribution of slit-cracks and the three-dimensional distribution of circular cracks. In Fig. 1 one can find some orientational distributions between the upon-described limiting cases, in 2D elasticity. The angular distribution of cracks is statistically well described by an order parameter P. Similarly, in Fig. 2 several orientational distributions of circular cracks in 3D elasticity have been shown. In such a case, two order parameters S and T define the orientational distribution of cracks. The mathematical details will be discussed later on.

We want to draw some comparisons between our approach and previous ones on this topic: for example in Refs. [12,13] different angular distributions of cracks have been considered but it is done by means of an *a priori* given parameterised probability density. In our approach the probability density is completely arbitrary and we analytically verify that only some expected values (the so called order parameters) contribute to modify the effective properties of the medium (under the sole hypothesis of low cracks density). For example, for the case of circular (penny shaped) cracks in Ref. [12,13] the authors adopt only one scatter parameter in the density function but here we verify that two order parameters are necessary to describe

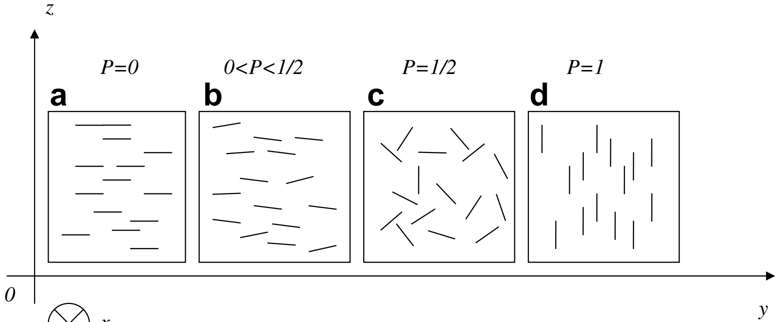

Fig. 1. Structure of a microcracked solid with slit-cracks in 2D elasticity. One can find some orientational distributions ranging from order to disorder. The order parameter P is indicated.

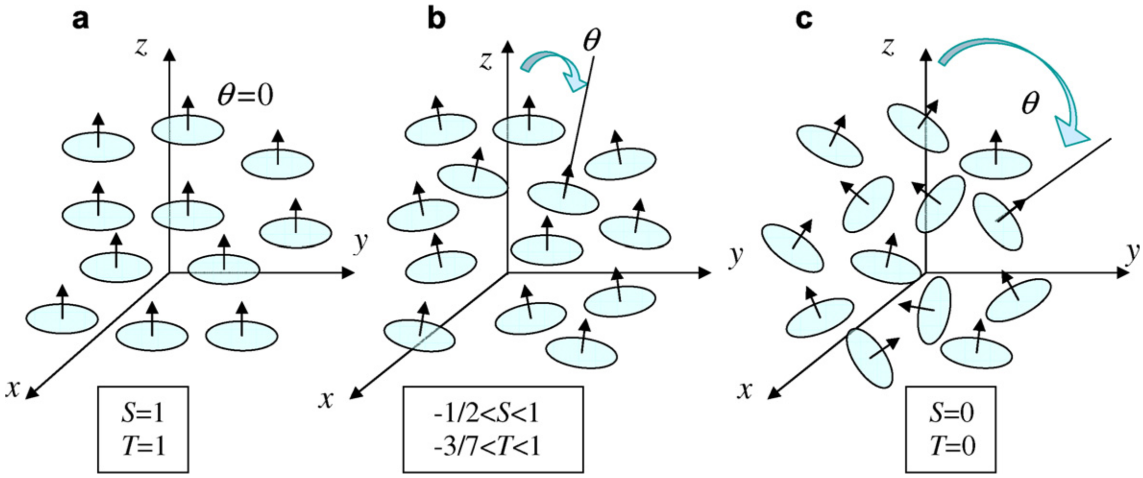

Fig. 2. Structure of a microcracked solid with circular cracks in 3D elasticity. One can find some orientational distributions ranging from order to disorder. The order parameters S and T are indicated.

an arbitrary distribution. Therefore, the peculiar character of the present work is given by the complete arbitrariness of the orientational distribution of cracks. This point is of vital importance to describe natural and artificial heterogeneous materials with complex microstructure. Moreover, we address also the analysis of a population of arbitrarily scattered parallel slit cracks, which is not, for the author's knowledge, present in the literature yet.

However, the results obtained may have several applications. For example an interesting topic is the study of microcracked rocks in geology: cracking originates in rocks from a number of geological processes, of which thermal gradients and tectonic stress are particularly important. Experiments on thermally-induced cracking and stress-induced cracking suggest that the former process produces a fairly isotropic distribution of predominantly intergranular cracks, while the latter produces a strongly anisotropic distribution of intragranular and transgranular cracks, with the majority of cracks oriented parallel to the direction of the maximum principal stress [14,15]. Therefore, the study of arbitrary distributions of cracks is very important from the geological point of view.

Another example is given by the application of these theories to biological materials: not isotropic fracture mechanisms have long been proposed for mineralized biological tissues like bone and dentin [16]. Moreover, not homogeneous distribution of cracks should be studied because site-specific accumulation of microcracks is considered a key factor that decreases the resistance of whole bones to fracture.

To conclude, this paper describes a theoretical-computational procedure that helps us to solve the following generic practical problem: often, in many applications of material science and in applied engineering it is important to accurately estimate the mechanical properties of a medium which is microcracked with an arbitrarily given angular distribution of cracks. The methodology, here introduced, allows us to evaluate the stiffness properties of the microcracked medium in terms of the orientational distribution of cracks inside the medium itself. As above said, such a result is obtained through the definition of some order parameters that reveal a broad and general applicability to many problems of great engineering significance.

2. Two-dimensional theory of pseudo-oriented cracks

This section deals with the analysis of distributions of two-dimensional cracks (slit-cracks) in isotropic solids. A single crack is modelled by means of a limiting process carried out on an ellipsoidal void: we consider an ellipsoid with semi-axes a_x, a_y and a_z ($a_x > a_y > a_z > 0$) aligned, respectively, along the axes x, y and z of a given reference frame. When one of the principal axes of the ellipsoidal void, say a_x, becomes infinite and the minor axis a_z becomes infinitely small, the ellipsoid becomes a slit-like crack.

The homogeneous solid matrix, before the damaging due to the cracks appearing, is characterised by the relation $T_{ij} = L^{1}_{ijkl}E_{kl}$ where $\mathbf{T}$ is the stress tensor (3×3 sized), $\mathbf{E}$ is the strain tensor (3×3 sized) and $\mathbf{L}^1$ is the constant stiffness tensor.

We start with some definitions used to simplify the problem. Instead of describing the strain with the complete symmetric tensor we adopt a vector, which contains the six independent elements in a given order (Voigt notation); the same approach is used for the stress (T means transposed):

$$\widehat{\mathbf{E}} = [E_{11} \quad E_{22} \quad E_{33} \quad E_{12} \quad E_{23} \quad E_{13}]^{\mathrm{T}}; \quad \widehat{\mathbf{T}} = [T_{11} \quad T_{22} \quad T_{33} \quad T_{12} \quad T_{23} \quad T_{13}]^{\mathrm{T}} \tag{1}$$

Adopting this notation scheme the stiffness four-index tensor for the isotropic solid is represented by a simpler matrix with six rows and six columns:

$$\widehat{\mathbf{L}}^1 = \begin{bmatrix} k+\frac{4}{3}\mu & k-\frac{2}{3}\mu & k-\frac{2}{3}\mu & 0 & 0 & 0 \\ k-\frac{2}{3}\mu & k+\frac{4}{3}\mu & k-\frac{2}{3}\mu & 0 & 0 & 0 \\ k-\frac{2}{3}\mu & k-\frac{2}{3}\mu & k+\frac{4}{3}\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \tag{2}$$

so that the stress-strain relation becomes $\widehat{\mathbf{T}} = \widehat{\mathbf{L}}^1\widehat{\mathbf{E}}$ in the matrix and $\widehat{\mathbf{T}} = 0$ inside each void miming a crack. We remember that instead of using the bulk modulus k and the shear modulus μ we may adopt the Young modulus E and the Poisson ratio v, defined as follows:

$$\left\{E = \frac{9k\mu}{\mu+3k}; \quad v = \frac{3k-2\mu}{2(\mu+3k)}\right\} \Longleftrightarrow \left\{\mu = \frac{E}{2(1+v)}; \quad k = \frac{E}{3(1-2v)}\right\} \tag{3}$$

These relations will be often used throughout all the paper. At this point, to begin the strain computation we take into consideration a single ellipsoidal inclusion (void) embedded in an isotropic matrix; to perform the computation we take into consideration a voids with a_x infinite and a_y and a_z finite and different from zero. In other words the void is an elliptic cylinder aligned with the x-axis; in a second phase we will carry out the limit of $a_z \to 0$ obtaining the flat inhomogeneity. We suppose that the matrix is placed in an equilibrated state of infinitesimal constant elastic strain by external loads and then the void is embedded into the matrix reaching a corresponding state of strain, which is well described by the Eshelby theory [17,18]. In particular it is important to notice that the internal strain is constant (all the entries are constant) if the external or bulk strain is constant. The Eshelby theory allows us to write down a relationship between the internal and original strain when they are constant (or uniform) in the space. Accordingly with the Eshelby theory [19,20] the relationship between the original external strain and the induced internal strain (for voids) is given by

$$\widehat{\mathbf{E}}_i = \{\mathbf{I} - \widehat{\mathbf{S}}\}^{-1}\widehat{\mathbf{E}}_0 \tag{4}$$

where $\mathbf{I}$ is the identity matrix with size 6×6, $\widehat{\mathbf{E}}_i$ is the internal strain, $\widehat{\mathbf{E}}_0$ is the original external strain and $\widehat{\mathbf{S}}$ is the Eshelby tensor, which depends on the aspect ratio $e = a_z/a_y$ and on the Poisson ratio v of the matrix. Here, we remember that $\widehat{\mathbf{S}}$, for elliptic cylinders, is given by [19]:

$$\widehat{\mathbf{S}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{ev}{(1+e)(1-v)} & \frac{1}{2(1-v)}\left[\frac{e^2+2e}{(1+e)^2} + \frac{e(1-2v)}{(1+e)}\right] & \frac{1}{2(1-v)}\left[\frac{e^2}{(1+e)^2} - \frac{e(1-2v)}{(1+e)}\right] & 0 & 0 & 0 \\ \frac{v}{(1+e)(1-v)} & \frac{1}{2(1-v)}\left[\frac{1}{(1+e)^2} - \frac{(1-2v)}{(1+e)}\right] & \frac{1}{2(1-v)}\left[\frac{1+2e}{(1+e)^2} + \frac{(1-2v)}{(1+e)}\right] & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{e}{1+e} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2(1-v)}\left[\frac{1+e^2}{(1+e)^2} + 1 - 2v\right] & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1+e} \end{bmatrix} \tag{5}$$

In the following developments we have to perform the limit of $a_z \to 0$, which corresponds to $e \to 0$. It is important to notice that the tensor $\{\mathbf{I} - \widehat{\mathbf{S}}\}^{-1}$ is singular when $e \to 0$; this fact well describes the singular behaviour of the strain in flat cracks and takes place a crucial role in the further developments. With the aim of analysing the behaviour of a mixture of pseudo-oriented cracks, we need to evaluate the average value of the internal strain inside the elliptic cylinder over all its possible orientations or rotations in agreement with the given orientational distribution. To perform this averaging over the rotations we name the original reference frame with the letter B and we consider another generic reference frame that is named with the letter F. The relation between these bases B and F is described by means of a generic rotation matrix $\mathbf{R}(\theta)$:

$$\mathbf{R}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{6}$$

The angle that defines the pseudo-orientational distribution is θ: it describes a simple rotation along the x-axis. Therefore the following relations hold on between the different frames: $\mathbf{E}_i^B = \mathbf{R}\mathbf{E}_i^F\mathbf{R}^\mathrm{T}$ for the internal strain and $\mathbf{E}_0^B = \mathbf{R}\mathbf{E}_0^F\mathbf{R}^\mathrm{T}$ for the bulk strain (here the subscript T means transposed). These expressions have been written with standard notation for the strain (3×3 sized matrix). They may be converted in our notation defining a matrix $\widehat{\mathbf{M}}(\theta)$, 6×6 sized, which acts as a rotation matrix on our strain vectors: so, we may write $\widehat{\mathbf{E}}_i^B = \widehat{\mathbf{M}}\widehat{E}_i^F$ inside the ellipsoid and $\widehat{\mathbf{E}}_0^B = \widehat{\mathbf{M}}\widehat{E}_0^F$ outside it. The entries of the matrix $\widehat{\mathbf{M}}$ are completely defined by the comparison between the relations $\mathbf{E}_i^B = \mathbf{R}\mathbf{E}_i^F\mathbf{R}^\mathrm{T}$ and $\widehat{\mathbf{E}}_i^B = \widehat{\mathbf{M}}\widehat{E}_i^F$ and by considering the notation adopted for the strain:

$$\widehat{\mathbf{M}}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\theta)^2 & 1-\cos(\theta)^2 & 0 & -2\cos(\theta)\sin(\theta) & 0 \\ 0 & 1-\cos(\theta)^2 & \cos(\theta)^2 & 0 & 2\cos(\theta)\sin(\theta) & 0 \\ 0 & 0 & 0 & \cos(\theta) & 0 & -\sin(\theta) \\ 0 & \cos(\theta)\sin(\theta) & -\cos(\theta)\sin(\theta) & 0 & -1+2\cos(\theta)^2 & 0 \\ 0 & 0 & 0 & \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \tag{7}$$

$$\widehat{\mathbf{M}}(\theta)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\theta)^2 & 1-\cos(\theta)^2 & 0 & 2\cos(\theta)\sin(\theta) & 0 \\ 0 & 1-\cos(\theta)^2 & \cos(\theta)^2 & 0 & -2\cos(\theta)\sin(\theta) & 0 \\ 0 & 0 & 0 & \cos(\theta) & 0 & \sin(\theta) \\ 0 & -\cos(\theta)\sin(\theta) & \cos(\theta)\sin(\theta) & 0 & -1+2\cos(\theta)^2 & 0 \\ 0 & 0 & 0 & -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \tag{8}$$

Eq. (4) is written on the frame B and therefore it actually reads $\widehat{\mathbf{E}}_i^B = \{\mathbf{I} - \widehat{\mathbf{S}}\}^{-1}\widehat{\mathbf{E}}_0^B$; this latter may be reformulated on the generic frame F, by using Eqs. (7) and (8), simply obtaining:

$$\widehat{\mathbf{E}}_i^F = \{\widehat{\mathbf{M}}(\theta)^{-1}[\mathbf{I} - \widehat{\mathbf{S}}]^{-1}\widehat{\mathbf{M}}(\theta)\}\widehat{\mathbf{E}}_0^F \tag{9}$$

As above said, at the end of this procedure we are interested in the limit of $e \to 0$ and therefore we may consider only terms in $1/e$ in the previous Eq. (9); all the other terms will be neglected for very small values of e. Therefore, we obtain the following result considering only the terms with $1/e$:

$$\widehat{\mathbf{M}}(\theta)^{-1}[\mathbf{I}-\widehat{\mathbf{S}}]^{-1}\widehat{\mathbf{M}}(\theta)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2v(1-v)(1-\cos(\theta)^2)}{e(1-2v)} & \frac{2(1-v)^2(1-\cos(\theta)^2)}{e(1-2v)} & \frac{2v(1-v)(1-\cos(\theta)^2)}{e(1-2v)} & 0 & \frac{2(1-v)\sin(\theta)\cos(\theta)}{e} & 0 \\ \frac{2v(1-v)\cos(\theta)^2}{e(1-2v)} & \frac{2v(1-v)\cos(\theta)^2}{e(1-2v)} & \frac{2(1-v)^2\cos(\theta)^2}{e(1-2v)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\cos(\theta)^2}{e} & 0 & 0 \\ \frac{2v(1-v)\sin(\theta)\cos(\theta)}{e(1-2v)} & \frac{(1-v)\sin(\theta)\cos(\theta)}{e(1-2v)} & \frac{(1-v)\sin(\theta)\cos(\theta)}{e(1-2v)} & 0 & \frac{1-v}{e} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\cos(\theta)^2}{e} \end{bmatrix} \tag{10}$$

So, Eqs. (9) and (10) furnish the relationship between the external strain and the internal one for a slit-crack with very small eccentricity e after a rotation of an angle θ in the z–y plane. We want to analyse the averaged effects of the orientational distribution of cracks and therefore we need to average Eq. (9) over all the possible orientation of a crack in the solid. Thus, the angle θ assumes, by hypotheses, the role of a random variable symmetrically distributed over the range $(-\pi/2,\pi/2)$. The symmetry of the probability density assures that the average value of $\sin(\theta)\cos(\theta)$ (appearing in Eq. (10)) is exactly zero and the result depends only on the average value of $\cos^2(\theta)$. So, we may define the following order parameter, which completely describes the state of order/disorder of the distribution of cracks:

$$P=\langle 1-\cos(\theta)^2\rangle \tag{11}$$

It is easy to observe that P assumes special values for particular angular distributions of cracks: if $P=0$ all the cracks are parallel to the y-axis (horizontal order), if $P=1$ all the cracks are parallel to the z-axis (vertical order) and if $P=1/2$ the angle of rotation is uniformly distributed in the range $(-\pi/2,\pi/2)$ leading to a state of complete disorder (2D isotropic medium). The other values cover all the orientational distribution between the random and the parallel ones (see Fig. 1 for some examples). Finally, the averaged value of Eq. (9) is given by:

$$\langle\widehat{\mathbf{E}}_i^F\rangle=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2v(1-v)P}{e(1-2v)} & \frac{2(1-v)^2P}{e(1-2v)} & \frac{2v(1-v)P}{e(1-2v)} & 0 & 0 & 0 \\ \frac{2v(1-v)(1-P)}{e(1-2v)} & \frac{2v(1-v)(1-P)}{e(1-2v)} & \frac{2(1-v)^2(1-P)}{e(1-2v)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{P}{e} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-v}{e} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-P}{e} \end{bmatrix}\widehat{\mathbf{E}}_0^F=\widehat{\mathbf{C}}\widehat{E}_0^F \tag{12}$$

or $\langle\widehat{\mathbf{E}}_i\rangle=\widehat{\mathbf{C}}\widehat{E}_0$ omitting the reference frame used in this context. In literature, the tensor $\widehat{\mathbf{C}}$ is the so-called averaged Wu's tensor [2]. As before, in the previous Eq. (12), only the terms with $1/e$ are maintained. Now we may analyse the actual distribution of cracks: we consider a region of the plan z–y having area A and N slit-cracks here uniformly dispersed with the angular distribution characterised by the order parameter P. Therefore, the volume fraction of the inclusions is given by$c=\pi a_z a_y N/A$ where a_z and a_y are linked by the relation $e=a_z/a_y$. We may compute the average value of the elastic strain over the whole cracked material by means of the relation:

$$\langle\widehat{\mathbf{E}}\rangle=(1-c)\widehat{\mathbf{E}}_0+c\langle\widehat{\mathbf{E}}_i\rangle=[(1-c)\mathbf{I}+c\widehat{\mathbf{C}}]\widehat{\mathbf{E}}_0 \tag{13}$$

where we have considered the average strain outside the inclusions approximately identical to the bulk strain $\widehat{\mathbf{E}}_0$ (hypothesis of low cracks density). In this approximation of non-interacting cracks, each crack is subjected to the same external load, unperturbed by the neighbours. This is a typical approximated scheme largely used in the field of the homogenisation techniques because it provides simple but significant results, which are valid only for low values of the volume fraction of the dispersed phases (in our case it means low cracks density).

In other words, we may say that we have adopted the dilute scheme, which is pertinent to the situations where one can consider that the cracks are not interacting with each other [4,8]. We understand that it is a limitation of applicability of all the results but it is an interesting way that we can follow in order to obtain results in closed form. Moreover, to generalise some of the results to higher values of the cracks density we have adopted, in the following sections, the iterated homogenisation scheme and the differential effective medium theory. However, we define $\widehat{\mathbf{L}}_{\mathrm{eq}}$ as the equivalent stiffness tensor of the whole mixture (which is anisotropic) by means of the relation $\langle\widehat{\mathbf{T}}\rangle = \widehat{\mathbf{L}}_{\mathrm{eq}}\langle\widehat{\mathbf{E}}\rangle$; to evaluate $\widehat{\mathbf{L}}_{\mathrm{eq}}$ we compute the average value $\langle\widehat{\mathbf{T}}\rangle$ of the stress inside the random material. We also define V as the total volume of the mixture, V_{e} as the total volume of the embedded cracks and V_0 as the volume of the remaining space among the inclusions (so that $V = V_{\mathrm{e}} \cup V_0$). The average value of $\widehat{\mathbf{T}} = \widehat{\mathbf{L}}(\bar{r})\widehat{\mathbf{E}}$ over the volume of the whole material is evaluated as follows ($\widehat{\mathbf{L}}(\bar{r}) = \widehat{\mathbf{L}}^1$ if $\bar{r} \in V_0$ and $\widehat{\mathbf{L}}(\bar{r}) = 0$ if $\bar{r} \in V_{\mathrm{e}}$):

$$\begin{aligned}\langle\widehat{\mathbf{T}}\rangle &= \frac{1}{V}\int_V \widehat{\mathbf{L}}(\bar{r})\widehat{\mathbf{E}}(\bar{r})\,\mathrm{d}\bar{r} = \frac{1}{V}\widehat{\mathbf{L}}^1\int_{V_0}\widehat{\mathbf{E}}(\bar{r})\,\mathrm{d}\bar{r} \\ &= \frac{1}{V}\widehat{\mathbf{L}}^1\int_{V_0}\widehat{\mathbf{E}}(\bar{r})\,\mathrm{d}\bar{r} + \frac{1}{V}\widehat{\mathbf{L}}^1\int_{V_{\mathrm{e}}}\widehat{\mathbf{E}}(\bar{r})\,\mathrm{d}\bar{r} - \frac{1}{V}\widehat{\mathbf{L}}^1\int_{V_{\mathrm{e}}}\widehat{\mathbf{E}}(\bar{r})\,\mathrm{d}\bar{r} = \widehat{\mathbf{L}}^1\langle\widehat{\mathbf{E}}\rangle - c\widehat{\mathbf{L}}^1\langle\widehat{\mathbf{E}}_i\rangle \\ &= \widehat{\mathbf{L}}^1\langle\widehat{\mathbf{E}}\rangle - c\widehat{\mathbf{L}}^1\widehat{\mathbf{C}}\widehat{\mathbf{E}}_0\end{aligned} \tag{14}$$

Drawing a comparison between Eqs. (13) and (14) we may find a complete expression, which allows us to estimate the equivalent stiffness tensor $\widehat{\mathbf{L}}_{\mathrm{eq}}$:

$$\widehat{\mathbf{L}}_{\mathrm{eq}} = \widehat{\mathbf{L}}^1\{\mathbf{I} - c\widehat{\mathbf{C}}[(1-c)\mathbf{I} + c\widehat{\mathbf{C}}]^{-1}\} \tag{15}$$

As above said, the volume fraction c is given by $c = \pi a_z a_y N/A$ or, remembering that $e = a_z/a_y$ and defining $a = a_y$ as the half-length of the slit-microcrack, by $c = \pi a^2 eN/A$. A characteristic quantity describing 2D microcracked solids is the following:

$$\alpha = \frac{\pi a^2}{A}N \tag{16}$$

From some approximated estimations we may deduce that a reasonable limit, for obtaining good results in this case of non-interacting slit-cracks, is given by $\alpha < 0.1$. It means, for example, that for a region with area 1 m^2 and cracks with half-length of 1 cm we may consider $N < 300$ approximately. So, we may write $c = \alpha e$; it means that the limit for exactly flat cracks is obtained with $e \to 0$ or equivalently for $c \to 0$. As one can see in Eq. (15), we are interested in the limit of the quantity $c\widehat{\mathbf{C}}$ when $e \to 0$: for following purposes we define $\lim_{e\to 0} c\widehat{\mathbf{C}} = \widehat{\mathbf{G}}$. Taking into consideration the definition of the tensor $\widehat{\mathbf{C}}$ given in Eq. (12) we immediately obtain the requested result:

$$\lim_{e\to 0} c\widehat{\mathbf{C}} = \widehat{\mathbf{G}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2\nu(1-\nu)P\alpha}{(1-2\nu)} & \frac{2(1-\nu)^2 P\alpha}{(1-2\nu)} & \frac{2\nu(1-\nu)P\alpha}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{2\nu(1-\nu)(1-P)\alpha}{(1-2\nu)} & \frac{2\nu(1-\nu)(1-P)\alpha}{(1-2\nu)} & \frac{2(1-\nu)^2(1-P)\alpha}{(1-2\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & P\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-\nu)\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-P)\alpha \end{bmatrix} \tag{17}$$

Furthermore, the exact limiting value for the stiffness tensor derives from Eqs. (15) and (17):

$$\widehat{\mathbf{L}}_{\mathrm{eq}} = \widehat{\mathbf{L}}^1\{\mathbf{I} - \widehat{\mathbf{G}}[\mathbf{I} + \widehat{\mathbf{G}}]^{-1}\} \tag{18}$$

where $\mathbf{L}^1$ is given by Eq. (2) and $\widehat{\mathbf{G}}$ is given by Eq. (17). It is evident by the microgeometry of the system that three unequal axes, at right angles to each other, characterise the solid: one is the x-axis that is the direction of alignment of the slit-cracks, the others are the axes y and z which have different elastic behaviour because of the pseudo-random orientation of the cracks. The axes y and z are equivalent from the mechanical point of view only if $P = 1/2$ and we obtain an overall transversely isotropic material. However, the three different behaviours along the three axes lead to an orthorhombic anisotropy for the whole system:

$$\widehat{\mathbf{L}}_{\mathrm{eq}} = \begin{bmatrix} L_{1111} & L_{1122} & L_{3311} & 0 & 0 & 0 \\ L_{1122} & L_{2222} & L_{2233} & 0 & 0 & 0 \\ L_{3311} & L_{2233} & L_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{3131} \end{bmatrix} \tag{19}$$

A solid with orthorhombic anisotropy is described by a stiffness tensor with nine independent parameters as one can see in Eq. (19). A long but straightforward application of Eq. (18) allows us to obtain the following closed form expressions for the stiffness tensor entries:

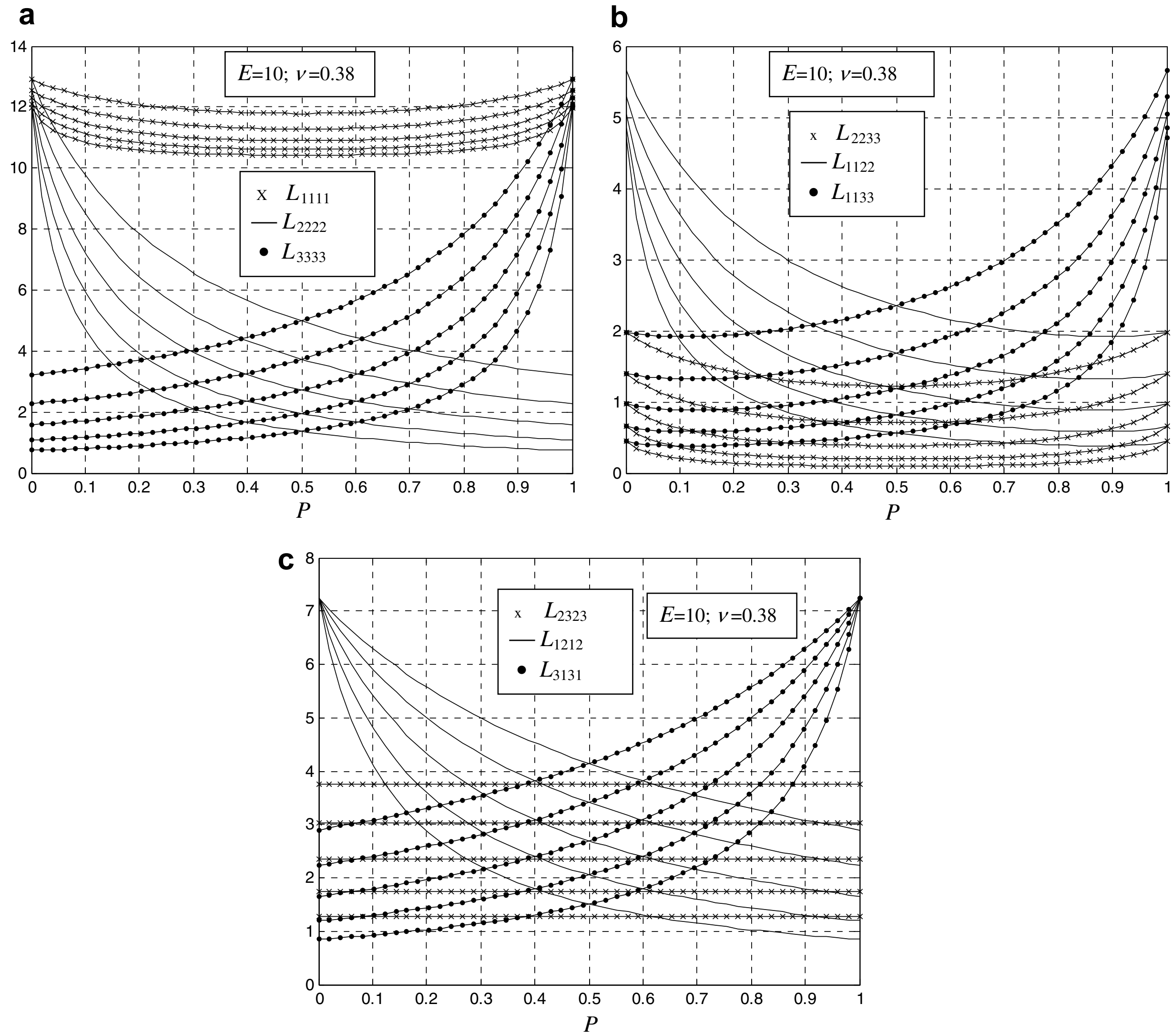

Fig. 3. The entries of the stiffness tensor for a 2D microcracked solid are represented. Results given by Eqs. (20)–(22) have been shown, respectively in (a)–(c). The nine elastic moduli have been represented versus the order parameter P using five values of the cracks density α: 1.5, 2.25, 3.37, 5.06 and 7.6.

$$\begin{cases} L_{1111} = \frac{4\alpha^2 P(1-P)(1+\nu)(1-\nu)^2+2(1-\nu)\alpha+1-\nu}{[4\alpha^2 P(1-P)(1-\nu)^2+2(1-\nu)^2\alpha+1-2\nu](1+\nu)} E \\ L_{2222} = \frac{(1-\nu)[1+2(1-P)\alpha]}{[4\alpha^2 P(1-P)(1-\nu)^2+2(1-\nu)^2\alpha+1-2\nu](1+\nu)} E \\ L_{3333} = \frac{(1-\nu)[1+2\alpha P]}{[4\alpha^2 P(1-P)(1-\nu)^2+2(1-\nu)^2\alpha+1-2\nu](1+\nu)} E \end{cases} \tag{20}$$

$$\begin{cases} L_{1122} = \frac{\nu[1+2(1-P)(1-\nu)\alpha]}{[4\alpha^2 P(1-P)(1-\nu)^2+2(1-\nu)^2\alpha+1-2\nu](1+\nu)} E \\ L_{2233} = \frac{\nu}{[4\alpha^2 P(1-P)(1-\nu)^2+2(1-\nu)^2\alpha+1-2\nu](1+\nu)} E \\ L_{3311} = \frac{\nu[1+2P(1-\nu)\alpha]}{[4\alpha^2 P(1-P)(1-\nu)^2+2(1-\nu)^2\alpha+1-2\nu](1+\nu)} E \end{cases} \tag{21}$$

$$\begin{cases} L_{1212} = \frac{1}{(1+\alpha P)(1+\nu)} E \\ L_{2323} = \frac{1}{[1+(1-\nu)\alpha](1+\nu)} E \\ L_{3131} = \frac{1}{[1+\alpha(1-P)](1+\nu)} E \end{cases} \tag{22}$$

Eqs. (20)–(22) represent the complete characterisation of a solid with a given distribution of slit-cracks under the hypothesis of low cracks density. In the following section we analyse some generalisations to higher values of the cracks density. It is interesting to observe that the following property holds on for these results: to change the order parameter P with $1-P$ corresponds to invert the axis y with the axis z. It means that we may change the index '2' with the index '3' and P with $1-P$ obtaining the same results. This is perfectly coherent with the physical meaning of the order parameter P: two different materials with P and $1-P$ have the same distribution of cracks but they are rotated of 90° in the plane y–z. Results given by Eqs. (20)–(22) have been shown in Fig. 3. The nine elastic moduli have been represented versus the order parameter P using five values of the quantity α: 1.5, 2.25, 3.37, 5.06 and 7.6.

3. 2D isotropic distribution of cracks: iterative homogenisation and differential schemes

When the slit-cracks are uniformly oriented in the y–z plane the overall microcracked material is transversely isotropic and the order parameter assumes the value $P = 1/2$; then, the corresponding stiffness tensor given by Eq. (19) reduces to the following:

$$\widehat{\mathbf{L}}_{\text{eq}} = \begin{bmatrix} n & l & l & 0 & 0 & 0 \\ l & k+m & k-m & 0 & 0 & 0 \\ l & k-m & k+m & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p & 0 & 0 \\ 0 & 0 & 0 & 0 & 2m & 0 \\ 0 & 0 & 0 & 0 & 0 & 2p \end{bmatrix} \tag{23}$$

where the five Hill parameters (typically used for transverse isotropy [3]) are given by Eqs. (20)–(22) with $P = 1/2$:

$$\begin{cases} n = \frac{[1+\alpha(1+\nu)](1-\nu)}{[1-2\nu+(1-\nu)\alpha](1+\nu)} E \\ l = \frac{\nu}{[1-2\nu+(1-\nu)\alpha](1+\nu)} E \\ k = \frac{1}{2[1-2\nu+(1-\nu)\alpha](1+\nu)} E \\ m = \frac{1}{2[1+(1-\nu)\alpha](1+\nu)} E \\ p = \frac{1}{(2+\alpha)(1+\nu)} E \end{cases} \tag{24}$$

Typically, in 2D elasticity a transversely isotropic medium may be used under the conditions of plane stress or plane strain. In this section we analyse the consequences of Eqs. (23) and (24) in such cases. We begin with the hypothesis of plane stress (on the plane y–z) and we observe that, in these conditions, a transversely isotropic material, defined by Eq. (23), corresponds to an isotropic one with these Young modulus and Poisson ratio:

$$\begin{cases} E_{\text{eq}} = \frac{4m(l^2-nk)}{l^2-nk-nm} \\ \nu_{\text{eq}} = \frac{l^2-nk+nm}{l^2-nk-nm} \end{cases} \tag{25}$$

These relations have been derived as follows: we take into consideration a transversely isotropic medium described by Eq. (23) and an isotropic medium described by E_{eq} and ν_{eq}. By imposing the plane stress conditions in both materials and drawing a comparison of the obtained results we may find exactly Eq. (25). However, by using Eq. (25) with the Hill moduli defined in Eq. (24) we obtain the equivalent elastic moduli in the plane stress case and their first order expansions in the parameter α:

$$\begin{cases} E_{\text{eq}} = \frac{E}{1+\alpha(1-\nu^2)} \cong E[1-\alpha(1-\nu^2)] \\ \nu_{\text{eq}} = \frac{\nu}{1+\alpha(1-\nu^2)} \cong \nu[1-\alpha(1-\nu^2)] \end{cases} \tag{26}$$

We remember that Eq. (26) holds true only for low values of the cracks density N/A that compares in the parameter α defined in Eq. (16). The first order expansions written in Eq. (26) are very useful to apply the iterated homogenisation method [6] that allows us to generalise the relations to higher values of the cracks density. The principles of this technique are here summarised: let's suppose that the effective moduli of a microcracked medium are known to be E_{eq} and ν_{eq}. Now, if a small additional number of cracks ΔN is created in the matrix, the change in the elastic moduli is approximated to be that which arise if the same infinitesimal number of cracks were added to a uniform, homogeneous matrix with moduli E_{eq} and ν_{eq}. This leads, when applied to Eq. (26), to the following difference equations, where the definition of the parameter α has been used:

$$\begin{cases} E_{\text{eq}}(N+\Delta N) = E_{\text{eq}}(N)\left[1-\frac{\pi a^2}{A}\Delta N(1-\nu_{\text{eq}}(N)^2)\right] \\ \nu_{\text{eq}}(N+\Delta N) = \nu_{\text{eq}}(N)\left[1-\frac{\pi a^2}{A}\Delta N(1-\nu_{\text{eq}}(N)^2)\right] \end{cases} \tag{27}$$

When the number of additional cracks ΔN assumes the role of an infinitesimal quantity the iterated homogenisation method converges to the differential effective medium theory [7,8] and the difference equations given in Eq. (27) became a pair of differential equations:

$$\begin{cases} \frac{\mathrm{d}E_{\text{eq}}}{\mathrm{d}N} = -\frac{\pi a^2}{A}(1-\nu_{\text{eq}}^2)E_{\text{eq}} \\ \frac{\mathrm{d}\nu_{\text{eq}}}{\mathrm{d}N} = -\frac{\pi a^2}{A}(1-\nu_{\text{eq}}^2)\nu_{\text{eq}} \end{cases} \tag{28}$$

They can be solved in closed form obtaining the final results for isotropic 3D elasticity in plane stress conditions:

$$\begin{cases} E_{\text{eq}} = \frac{E}{\sqrt{\nu^2+(1-\nu^2)e^{2\alpha}}} \\ \nu_{\text{eq}} = \frac{\nu}{\sqrt{\nu^2+(1-\nu^2)e^{2\alpha}}} \end{cases} \tag{29}$$

A similar analysis can be conducted for the plane strain case. Now, the transversely isotropic material, defined by Eq. (23), corresponds to an isotropic one with Young modulus and Poisson ratio given by

$$\begin{cases} E_{\text{eq}} = \frac{m}{k}(3k-m) \\ \nu_{\text{eq}} = \frac{k-m}{2k} \end{cases} \tag{30}$$

Moreover, by using Eq. (30) with the Hill moduli defined in Eq. (24) we obtain the equivalent elastic moduli under the plane strain condition and their first order expansions in the parameter α:

$$\begin{cases} E_{\text{eq}} = E\frac{[1+\nu+\alpha(1-\nu)]}{[1+\alpha(1-\nu)]^2(1+\nu)} \cong E\left[1-\alpha\frac{(1-\nu)(1+2\nu)}{1+\nu}\right] \\ \nu_{\text{eq}} = \frac{\nu}{1+\alpha(1-\nu)} \cong \nu[1-\alpha(1-\nu)] \end{cases} \tag{31}$$

As before the knowledge of the first order expansions is useful to apply the iterated homogenisation method, which leads, at the end of the procedure, to the following differential equations:

$$\begin{cases} \dfrac{\mathrm{d}E_{\mathrm{eq}}}{\mathrm{d}N} = -\dfrac{\pi a^2}{A} \dfrac{(1-\nu_{\mathrm{eq}})(1+2\nu_{\mathrm{eq}})}{1+\nu_{\mathrm{eq}}} E_{\mathrm{eq}} \\ \dfrac{\mathrm{d}\nu_{\mathrm{eq}}}{\mathrm{d}N} = -\dfrac{\pi a^2}{A}(1-\nu_{\mathrm{eq}})\nu_{\mathrm{eq}} \end{cases} \tag{32}$$

The solutions can be analytically obtained:

$$\begin{cases} E_{\mathrm{eq}} = E \dfrac{2\nu + (1-\nu)e^{\alpha}}{[\nu + (1-\nu)e^{\alpha}]^2(1+\nu)} \\ \nu_{\mathrm{eq}} = \dfrac{\nu}{\nu + (1-\nu)e^{\alpha}} \end{cases} \tag{33}$$

They represent the elastic moduli of an isotropic microcracked material under plane strain conditions. It is interesting to observe that our solutions (given by Eq. (29) for plane stress and by Eq. (33) for plane strain) depend exponentially on the cracks density. This fact explains the strong and speed damaging of a medium with an increasing number of cracks in a given region. In Fig. 4 these results have been shown versus the cracks density α: here a comparison with the 3D case, described in the following section, has been drawn.

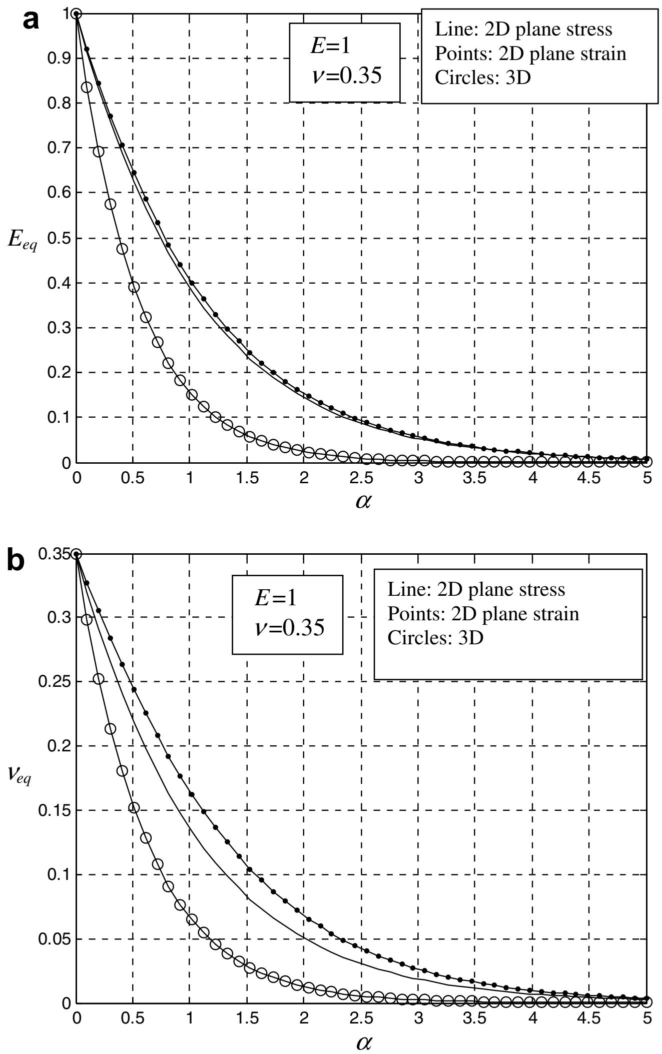

Fig. 4. The equivalent elastic moduli for an isotropic cracked solid are shown versus the cracks density α. A comparison among three cases is given: 2D plane stress (Eq. (29)), 2D plane strain (Eq. (33)) and 3D (Eq. (52)).

4. Three-dimensional distributions of circular cracks

In this section we are dealing with 3D distributions of circular planar cracks in isotropic solids (N cracks dispersed in a region with volume V). As before the main feature of this analysis is given by the pseudo-random orientation of the cracks inside the solid (see Fig. 2 for details). We consider a given orthonormal reference frame and we take as preferential direction of alignment the z-axis. Each crack embedded in the matrix is not completely random oriented. The overall medium has a positional disorder but a partial orientational order and it exhibits a uniaxial behaviour. The orientation of a crack is described by the following statistical rule: the principal axis (the normal direction) of each crack forms with the z-axis an angle θ, which follows a given probability density, $f(\theta)$ defined in $[0\ \pi]$ (see Fig. 2). The orientation of each crack is statistically independent from the orientation of the other ones. If $f(\theta) = \delta(\theta)$ (where δ is the Dirac delta function) we have all the cracks with $\theta = 0$ and therefore they are all oriented with the z-axis. If $f(\theta) = (1/2)\sin\theta$ all the cracks are uniformly random oriented in the space over all the possible orientations. Any other statistical distributions $f(\theta)$ define a transversely isotropic (uniaxial) material. In this section we develop a complete analysis of the effects of the state of order/disorder. This analysis allows us to evaluate the overall elastic properties of the microcracked material. From the point of view of the state of order we verified the following property: the elastic moduli of the material depend on the state of order through two parameters that are defined as $S = \langle P_2(\cos\theta)\rangle_\theta$ and $T = \langle P_4(\cos\theta)\rangle_\theta$. They correspond to the average values of the Legendre polynomial of order two and four, computed by means of the density probability $f(\theta)$. To begin, we take into consideration the Eshelby tensor of an ellipsoid of rotation ($a_x = a_y$) with the principal axis aligned along the z-axis of the reference frame; we define the aspect ratio e as $e = a_z/a_x = a_z/a_y$ where a_x, a_y and a_z are the semi-axes aligned, respectively, along the axes x, y and z of the given reference frame. The general structure of such a tensor is given by [19]:

$$\widehat{\mathbf{S}} = \begin{bmatrix} s_{1111} & s_{1122} & s_{1133} & 0 & 0 & 0 \\ s_{1122} & s_{1111} & s_{1133} & 0 & 0 & 0 \\ s_{3311} & s_{3311} & s_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{1111} - s_{1122} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{1313} \end{bmatrix} \tag{34}$$

Here the symmetries are evident and correctly describe the ellipsoid of rotation which has two equivalent axes and a third one with different behaviour. In Table 1 one can find the complete expressions of all the entries of the tensor defined in Eq. (34). The depolarisation factor L may be computed in closed form as follows and the result depends on the shape of the ellipsoid; it is prolate (of ovary or elongated form) if $e > 1$ and oblate (of planetary or flattened form) if $e < 1$ [19,21]:

Table 1
List of the complete expressions of all the entries of the Eshelby tensor defined in Eq. (34)

s_{1111}	$\frac{1}{8}\frac{-3e^2 + 13L - 4e^2L + 8Lve^2 - 8Lv}{(e^2-1)(-1+v)}$
s_{1122}	$-\frac{1}{8}\frac{e^2 + L - 4e^2L + 8Lve^2 - 8Lv}{(e^2-1)(-1+v)}$
s_{1133}	$-\frac{1}{2}\frac{2e^2L - e^2 + L + 2Lve^2 - 2Lv}{(e^2-1)(-1+v)}$
s_{3311}	$\frac{1}{2}\frac{-L + e^2 - 2e^2L - 2ve^2 + 2v + 4Lve^2 - 4Lv}{(e^2-1)(-1+v)}$
s_{3333}	$-\frac{2e^2 - 1 - 4e^2L + L - ve^2 + v + 2Lve^2 - 2Lv}{(e^2-1)(-1+v)}$
s_{1313}	$-\frac{1}{2}\frac{e^2L + 2L - 1 + Lve^2 - Lv - ve^2 + v}{(e^2-1)(-1+v)}$

$$L = \frac{e}{2}\int_0^{+\infty}\frac{\mathrm{d}\xi}{(\xi+1)^2(\xi+e^2)^{1/2}} = \begin{cases} \frac{e}{4(\sqrt{e^2-1})^3}\left[2e\sqrt{e^2-1}+\ln\frac{e-\sqrt{e^2-1}}{e+\sqrt{e^2-1}}\right] & \text{if } e>1 \\ \frac{e}{4(\sqrt{1-e^2})^3}\left[\pi-2e\sqrt{1-e^2}-2\mathrm{arctg}\frac{e}{\sqrt{1-e^2}}\right] & \text{if } e<1 \end{cases} \tag{35}$$

With the aim of modelling a circular crack we will use the limit of $e \to 0$ (strongly oblate ellipsoid). As described in previous sections, the relationship between the original external strain and the induced internal strain (for voids) is given by $\widehat{\mathbf{E}}_i = \{\mathbf{I}-\widehat{\mathbf{S}}\}^{-1}\widehat{\mathbf{E}}_0$ [19,20]. We are interested in strongly oblate ellipsoidal voids and therefore we take into consideration only the terms with $1/e$ computing the requested inverse matrix (we have used Eq. (34) and the expressions listed in Table 1):

$$\{\mathbf{I}-\widehat{\mathbf{S}}\}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4\nu(1-\nu)}{\pi(1-2\nu)e} & \frac{4\nu(1-\nu)}{\pi(1-2\nu)e} & \frac{4(1-\nu)^2}{\pi(1-2\nu)e} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4(1-\nu)}{\pi(2-\nu)e} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{4(1-\nu)}{\pi(2-\nu)e} \end{bmatrix} \tag{36}$$

The limit of $e \to 0$ will be performed in successive phases. We now need to evaluate the average value of the internal strain inside the ellipsoid over all its possible orientations or rotations in the space (in agreement with the given orientational distribution). To perform this averaging over the rotations we name the original reference frame with the letter B and we consider another generic reference frame that is named with the letter F.

The relation between these bases B and F is described by means of a generic rotation matrix $\mathbf{R}(\psi,\theta,\varphi)$ where ψ, θ and φ are the Euler angles; we may consider this matrix as the product of three elementary rotations along the axes z, x and z, respectively:

$$\mathbf{R}(\psi,\theta,\varphi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{37}$$

The angle that defines the pseudo-orientational distribution is θ. Therefore the following relations hold on between the different frames: $\mathbf{E}_i^B = \mathbf{R}\mathbf{E}_i^F\mathbf{R}^{\mathrm{T}}$ for the internal strain and $\mathbf{E}_0^B = \mathbf{R}\mathbf{E}_0^F\mathbf{R}^{\mathrm{T}}$ for the bulk strain (here the subscript T means transposed). These expressions have been written with standard notation for the strain (3×3 sized matrix). They may be converted in our notation defining a matrix $\widehat{\mathbf{M}}(\psi,\theta,\varphi)$, 6×6 sized, which acts as a rotation matrix on our strain vectors: so, we may write $\widehat{\mathbf{E}}_i^B = \widehat{\mathbf{M}}\widehat{E}_i^F$ inside the ellipsoid and $\widehat{\mathbf{E}}_0^B = \widehat{\mathbf{M}}\widehat{E}_0^F$ outside it. The entries of the matrix $\widehat{\mathbf{M}}$ are completely defined by the comparison between the relations $\mathbf{E}_i^B = \mathbf{R}\mathbf{E}_i^F\mathbf{R}^{\mathrm{T}}$ and $\widehat{\mathbf{E}}_i^B = \widehat{\mathbf{M}}\widehat{E}_i^F$ and by considering the notation adopted for the strain. The relation between bulk strain and internal strain is written on the frame B and therefore it actually reads $\widehat{\mathbf{E}}_i^B = \{\mathbf{I}-\widehat{\mathbf{S}}\}^{-1}\widehat{\mathbf{E}}_0^B$; this latter may be reformulated on the generic frame F simply obtaining:

$$\widehat{\mathbf{E}}_i^F = \{\widehat{\mathbf{M}}(\Psi,\theta,\varphi)^{-1}\{\mathbf{I}-\widehat{\mathbf{S}}\}^{-1}\widehat{\mathbf{M}}(\Psi,\theta,\varphi)\}\widehat{\mathbf{E}}_0^F \tag{38}$$

The first average value of the strain inside the inclusion may be computed by means of the integration over all the possible rotations of the angles φ and ψ (they are uniformly distributed over the entire range $[0,2\pi]$). Then, we may perform the second averaging over the angle θ described by an arbitrary probability density $f(\theta)$ defined on the range $[0,\pi]$:

$$\langle\widehat{\mathbf{E}}_i\rangle_{\Psi,\varphi,\theta} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi}\{\widehat{\mathbf{M}}(\Psi,\theta,\varphi)^{-1}\{\mathbf{I}-\widehat{\mathbf{S}}\}^{-1}\widehat{\mathbf{M}}(\Psi,\theta,\varphi)\}f(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}\Psi\widehat{\mathbf{E}}_0 = \widehat{\mathbf{C}}\widehat{\mathbf{E}}_0 \tag{39}$$

A tedious but straightforward integration leads to the following general structure for the averaging tensor $\widehat{\mathbf{C}}$:

$$\widehat{\mathbf{C}} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ c_{1122} & c_{1111} & c_{1133} & 0 & 0 & 0 \\ c_{3311} & c_{3311} & c_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{1111}-c_{1122} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{1313} \end{bmatrix} \tag{40}$$

In Table 2 one can find the expressions of all the entries of the tensor $\widehat{\mathbf{C}}$defined in Eq. (40). We wish to point out that the expressions given in Table 2 are extremely convenient to perform the micromechanical averaging because it removes the problem of the integral evaluation and allows us to obtain results in closed form. In Ref. [21] one can find a complete description of the angular averaging procedure for anisotropic distribution of inclusions. We have defined two order parameters S and T as follows:

$$S = \langle P_2(\cos\theta)\rangle_\theta = \left\langle \frac{3}{2}\cos^2(\theta) - \frac{1}{2}\right\rangle_\theta = \int_0^\pi \left(\frac{3}{2}\cos^2(\theta) - \frac{1}{2}\right) f(\theta)\,\mathrm{d}\theta \tag{41}$$

$$T = \langle P_4(\cos\theta)\rangle_\theta = \left\langle \frac{35}{8}\cos^4(\theta) - \frac{15}{4}\cos^2(\theta) + \frac{3}{8}\right\rangle_\theta = \int_0^\pi \left(\frac{35}{8}\cos^4(\theta) - \frac{15}{4}\cos^2(\theta) + \frac{3}{8}\right) f(\theta)\,\mathrm{d}\theta \tag{42}$$

The two order parameters S and T defined in Eqs. (41) and (42) are subjected to the following constraints: $-1/2 < S < 1$ and $-3/7 < T < 1$. A point in the S–T plane, as indicated in Fig. 5, represents the degree of orientational order. Three particular cases of state of order can be taken into consideration: if $S = T = 1$ we are in the state of order (cracks with normal unit vectors aligned along the z-axis), if $S = T = 0$ we are in the state of disorder (cracks randomly oriented) and, finally, if $S = -1/2$ and $T = 3/8$ all crack normal unit vectors are lying randomly in planes perpendicular to the z-axis [21]. Now, we may observe that Eq. (15), obtained for the two-dimensional case, continues to be valid in the present three-dimensional study. Here the volume fraction c

Table 2
List of the complete expressions of all the entries of the tensor $\widehat{\mathbf{C}}$ defined in Eq. (40)

c_{1111}	$\frac{4}{105}\frac{(-1+\nu)(70S-100S\nu+25S\nu^2+9T\nu-18T\nu^2-70+91\nu-7\nu^2)}{e\pi(-2+\nu)(-1+2\nu)}$
c_{1122}	$-\frac{4}{35}\frac{\nu(-1+\nu)(5S\nu-20S-T+2T\nu-7\nu+21)}{e\pi(-2+\nu)(-1+2\nu)}$
c_{1133}	$-\frac{4}{35}\frac{\nu(-1+\nu)(-25S+15S\nu+4T-8T\nu-7\nu+21)}{e\pi(-2+\nu)(-1+2\nu)}$
c_{3311}	$\frac{4}{35}\frac{\nu(-1+\nu)(-45S+20S\nu-4T+8T\nu+7\nu+21)}{e\pi(-2+\nu)(-1+2\nu)}$
c_{3333}	$-\frac{4}{105}\frac{(-1+\nu)(140S-200S\nu+50S\nu^2-24T\nu+48T\nu^2+70-91\nu+7\nu^2)}{e\pi(-2+\nu)(-1+2\nu)}$
c_{1313}	$-\frac{4}{105}\frac{(-24T\nu+10S\nu+14\nu-35S-70)(-1+\nu)}{e\pi(-2+\nu)}$

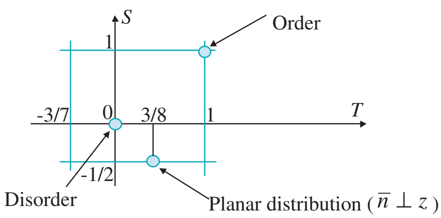

Fig. 5. The degree of orientational order is represented by a point in the S–T plane (3D case). Three particular cases of state of order can be observed: if $S = T = 1$ we are in the state of order, if $S = T = 0$ we are in the state of disordered and if $S = -1/2$ and $T = 3/8$ all cracks have normal vector lying randomly in planes perpendicular to the z-axis.

is given by $c = 4\pi a_x^2 a_z N/(3V)$ or, remembering that $e = a_z/a_x = a_z/a_y$ and defining $a = a_x = a_y$ as the radius of the circular microcrack, by $c = 4\pi a^3 eN/(3V)$. A characteristic quantity describing 3D microcracked solids is the following:

$$\alpha = \frac{a^3}{V} N \tag{43}$$

So, we may write $c = 4\pi\alpha e/3$; it means that the limit for exactly flat cracks is obtained with $e \to 0$ or with $c \to 0$. As one can see in Eq. (15) we are interested in the limit of the quantity $c\widehat{\mathbf{C}}$ when $e \to 0$. As before, we thus define $\lim_{e\to 0} c\widehat{\mathbf{C}} = \widehat{\mathbf{G}}$. Taking into consideration the definition of the tensor $\widehat{\mathbf{C}}$, given in Eq. (40) and Table 2, we immediately obtain the requested result:

$$\lim_{e\to 0} c\widehat{\mathbf{C}} = \widehat{\mathbf{G}} = \begin{bmatrix} g_{1111} & g_{1122} & g_{1133} & 0 & 0 & 0 \\ g_{1122} & g_{1111} & g_{1133} & 0 & 0 & 0 \\ g_{3311} & g_{3311} & g_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{1111} - g_{1122} & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{1313} \end{bmatrix} \tag{44}$$

In Table 3 the g_{ijkl} entries are listed and each of them is expressed as the sum of three contributes: the first one depending on the order parameter S, the second one depending on the order parameter T and the third one not depending on S and T represents the results for random oriented cracks ($S = T = 0$). In other words the first two terms represent the perturbation to the third one when the orientational distribution is different from the uniform one. So, results in Table 3 are the main achievement of this section, concerning with the characterisation of a 3D microcracked material.

Finally, it is a very long but straightforward task to verify that the general form of $\widehat{\mathbf{L}}_{eq}$ (deduced from Eq. (18)) is given by the following expression, in perfect agreement with transversely isotropic composites:

$$\widehat{\mathbf{L}}_{eq} = \begin{bmatrix} k+m & k-m & l & 0 & 0 & 0 \\ k-m & k+m & l & 0 & 0 & 0 \\ l & l & n & 0 & 0 & 0 \\ 0 & 0 & 0 & 2m & 0 & 0 \\ 0 & 0 & 0 & 0 & 2p & 0 \\ 0 & 0 & 0 & 0 & 0 & 2p \end{bmatrix} \tag{45}$$

The complete expressions of the corresponding Hill parameters are very complicated and therefore they are not very useful to better understand the physics of the microcracking process. It is instead interesting to evaluate these parameters up to the first order in the cracks density α:

Table 3
List of the complete expressions of all the entries of the tensor $\widehat{\mathbf{G}}$ defined in Eq. (44) in terms of the order parameters S and T

g_{1111}	$\frac{16(1-\nu)(20\nu-14-5\nu^2)\alpha S}{63(2-\nu)(1-2\nu)} - \frac{16\nu(1-\nu)\alpha T}{35(2-\nu)} + \frac{16(1-\nu)(10-13\nu+\nu^2)\alpha}{45(2-\nu)(1-2\nu)}$
g_{1122}	$\frac{16\nu(1-\nu)(\nu-4)\alpha S}{21(2-\nu)(1-2\nu)} - \frac{16\nu(1-\nu)\alpha T}{105(2-\nu)} + \frac{16\nu(1-\nu)(3-\nu)\alpha}{15(2-\nu)(1-2\nu)}$
g_{1133}	$\frac{16\nu(1-\nu)(3\nu-5)\alpha S}{21(2-\nu)(1-2\nu)} + \frac{64\nu(1-\nu)\alpha T}{105(2-\nu)} + \frac{16\nu(1-\nu)(3-\nu)\alpha}{15(2-\nu)(1-2\nu)}$
g_{3311}	$\frac{16\nu(1-\nu)(9-4\nu)\alpha S}{21(2-\nu)(1-2\nu)} + \frac{64\nu(1-\nu)\alpha T}{105(2-\nu)} + \frac{16\nu(1-\nu)(3-\nu)\alpha}{15(2-\nu)(1-2\nu)}$
g_{3333}	$\frac{32(1-\nu)(14-20\nu+5\nu^2)\alpha S}{63(2-\nu)(1-2\nu)} - \frac{128\nu(1-\nu)\alpha T}{105(2-\nu)} + \frac{16(1-\nu)(10-13\nu+\nu^2)\alpha}{45(2-\nu)(1-2\nu)}$
g_{1313}	$\frac{16(1-\nu)(7-2\nu)\alpha S}{63(2-\nu)} + \frac{128\nu(1-\nu)\alpha T}{105(2-\nu)} + \frac{32(1-\nu)(5-\nu)\alpha}{45(2-\nu)}$

$$
\begin{cases}
n = \dfrac{(1-\nu)E}{(1-2\nu)(1+\nu)} + \dfrac{16}{45}\dfrac{(1-\nu)(7\nu^3-32\nu^2+23\nu-10)\alpha E}{(1-2\nu)^2(2-\nu)(1+\nu)} + \dfrac{64}{63}\dfrac{(1-\nu)(\nu^2+3\nu-7)\alpha SE}{(1-2\nu)(2-\nu)(1+\nu)} + \dfrac{128}{105}\dfrac{\nu(1-\nu)\alpha TE}{(2-\nu)(1+\nu)} + O(\alpha^2) \\
l = \dfrac{\nu E}{(1-2\nu)(1+\nu)} + \dfrac{16}{45}\dfrac{\nu(1-\nu)(16\nu-19-\nu^2)\alpha E}{(1-2\nu)^2(2-\nu)(1+\nu)} + \dfrac{16}{63}\dfrac{\nu(1-\nu)(5\nu-13)\alpha SE}{(1-2\nu)(2-\nu)(1+\nu)} - \dfrac{64}{105}\dfrac{\nu(1-\nu)\alpha TE}{(2-\nu)(1+\nu)} + O(\alpha^2) \\
k = \dfrac{E}{2(1-2\nu)(1+\nu)} + \dfrac{16}{45}\dfrac{(1-\nu)(3\nu^3-8\nu^2+2\nu-5)\alpha E}{(1-2\nu)^2(2-\nu)(1+\nu)} + \dfrac{16}{63}\dfrac{(1-\nu)(7+10\nu-6\nu^2)\alpha SE}{(1-2\nu)(2-\nu)(1+\nu)} + \dfrac{32}{105}\dfrac{\nu(1-\nu)\alpha TE}{(2-\nu)(1+\nu)} + O(\alpha^2) \\
m = \dfrac{E}{2(1+\nu)} + \dfrac{16}{45}\dfrac{(1-\nu)(\nu-5)\alpha E}{(2-\nu)(1+\nu)} + \dfrac{16}{63}\dfrac{(1-\nu)(7-2\nu)\alpha SE}{(2-\nu)(1+\nu)} + \dfrac{16}{105}\dfrac{\nu(1-\nu)\alpha TE}{(2-\nu)(1+\nu)} + O(\alpha^2) \\
p = \dfrac{E}{2(1+\nu)} + \dfrac{16}{45}\dfrac{(1-\nu)(\nu-5)\alpha E}{(2-\nu)(1+\nu)} + \dfrac{8}{63}\dfrac{(1-\nu)(2\nu-7)\alpha SE}{(2-\nu)(1+\nu)} - \dfrac{64}{105}\dfrac{\nu(1-\nu)\alpha TE}{(2-\nu)(1+\nu)} + O(\alpha^2)
\end{cases}
\tag{46}
$$

In each of these expressions the terms have been ordered with the following rule: the first term represents the Hill modulus for an isotropic solid without cracks, the second term represents the perturbation due to

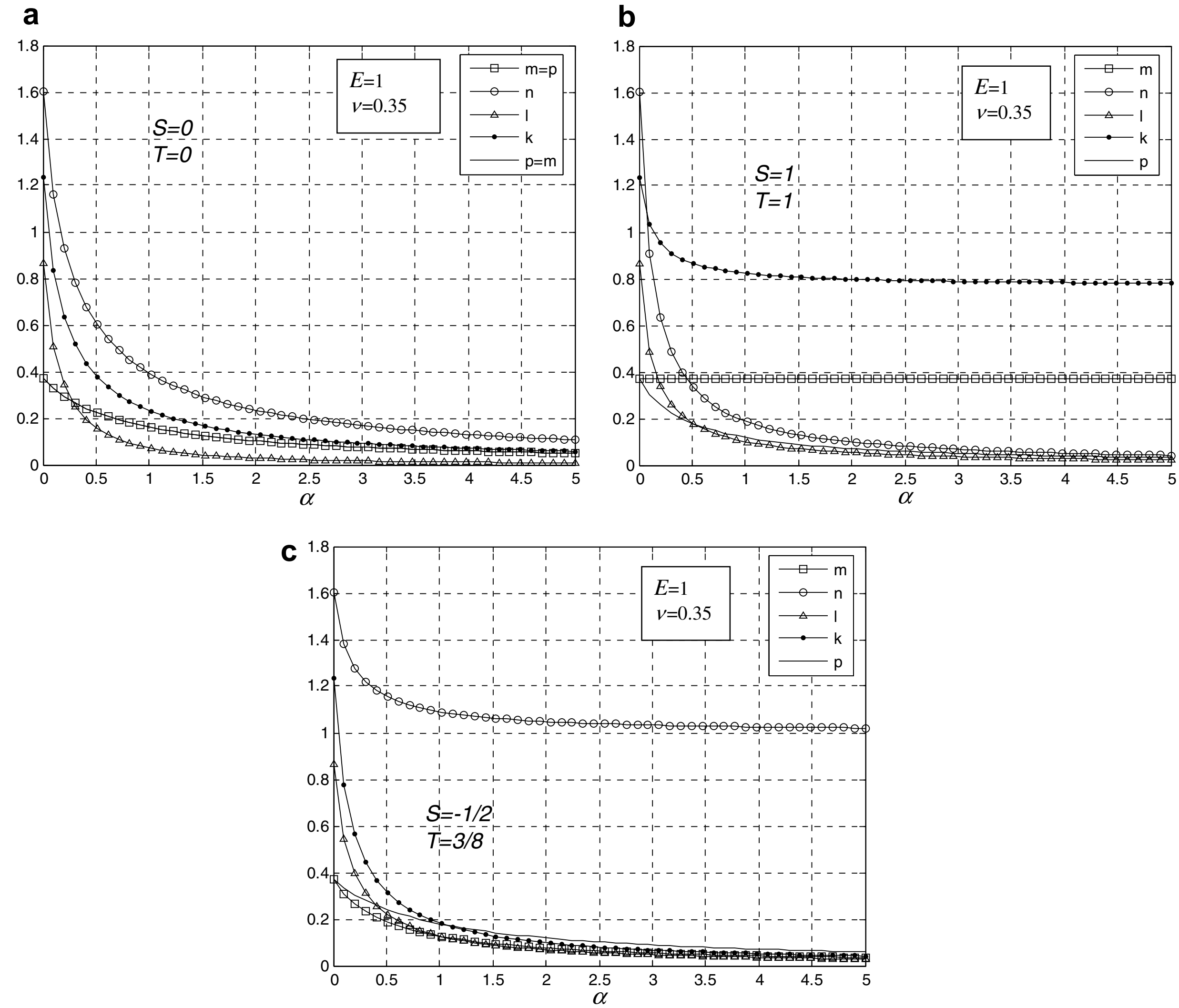

Fig. 6. Several results for the Hill parameters are reported in various states of order/disorder (3D case). Each plot corresponds to the indicated couple of order parameters S and T. In all cases we have considered the same solid ($E = 1$, $\nu = 0.35$) and we have shown the elastic moduli versus the cracks density.

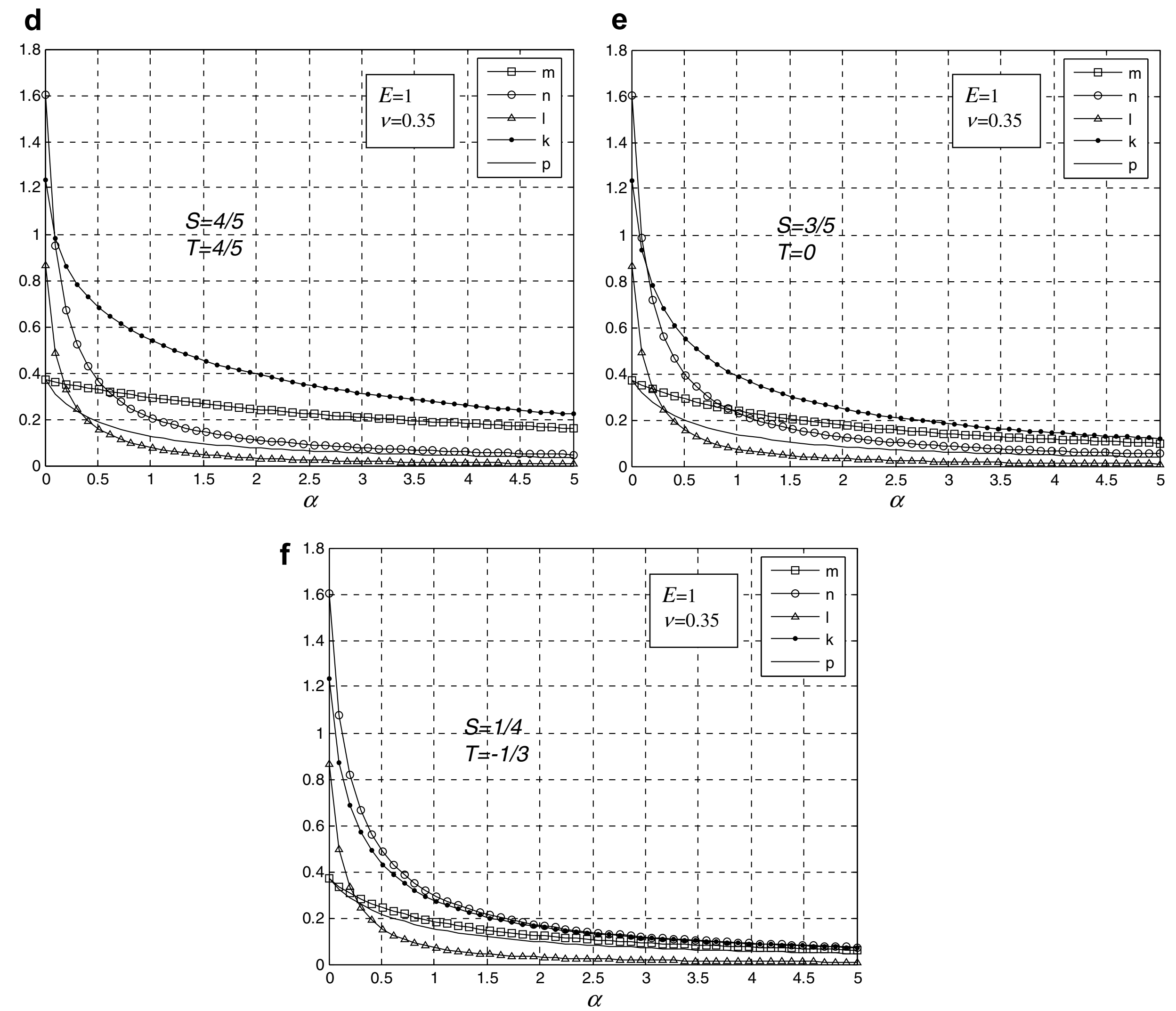

Fig. 6 (*continued*)

isotropic cracking ($S = T = 0$) of the material and the last two terms represent two additional perturbations introduced to take into account the particular angular distribution described by S and T. Approximated expressions given in Eq. (46) hold on only for a very small cracks density and thus for very small values of the parameter α. Of course, the exact procedure can be numerically implemented by means of Eq. (18) and the expressions listed in Table 3. We have developed a software code that implements such a procedure furnishing the five elastic moduli of the overall microcracked material. In Fig. 6 one can find several simulations describing different states of order of the material. For each orientational distribution (fixed S and T) we have plotted the five Hill parameters versus the parameter α. We have used an isotropic medium with $E = 1$ and $\nu = 0.35$ and a cracks density ranging from $\alpha = 0$ to $\alpha = 5$. This procedure allows us to understand more accurately the effects of a given angular distribution of cracks on the macroscopic behaviour of the medium under consideration.

Finally, it is interesting to write down results obtained with the complete procedure for some special cases, which may be useful in practical applications: if $S = T = 1$ we are dealing with a distribution of cracks aligned with the normal vectors along the z-axis; in this case the basic Eq. (18) and the expressions listed in Table 3 lead exactly to the following:

$$\begin{cases} n = \frac{3(1-\nu)}{[3(1-2\nu)+16(1-\nu)^2\alpha](1+\nu)}E \\ l = \frac{3\nu}{[3(1-2\nu)+16(1-\nu)^2\alpha](1+\nu)}E \\ k = \frac{16(1-\nu)^2\alpha+3}{2[3(1-2\nu)+16(1-\nu)^2\alpha](1+\nu)}E \\ m = \frac{1}{2(1+\nu)}E \\ p = \frac{3(2-\nu)}{2[3(2-\nu)+16(1-\nu)\alpha](1+\nu)}E \end{cases} \tag{47}$$

Moreover, if $S = -1/2$ and $T = 3/8$ we are dealing with a distribution of cracks randomly oriented but having the normal vectors perpendicular to the z-axis; as before, the basic Eq. (18) and the expressions listed in Table 3 lead exactly to the following:

$$\begin{cases} n = \frac{(1-\nu)[8(1+\nu)\alpha+3]}{[3(1-2\nu)+8(1-\nu)\alpha](1+\nu)}E \\ l = \frac{3\nu}{[3(1-2\nu)+8(1-\nu)\alpha](1+\nu)}E \\ k = \frac{3}{2[3(1-2\nu)+8(1-\nu)\alpha](1+\nu)}E \\ m = \frac{3(2-\nu)}{2[3(2-\nu)+4(1-\nu)(4-\nu)\alpha](1+\nu)}E \\ p = \frac{3(2-\nu)}{2[3(2-\nu)+8(1-\nu)\alpha](1+\nu)}E \end{cases} \tag{48}$$

Finally, with an isotropic dispersion of cracks with $S = T = 0$ we obtain an overall isotropic behaviour of the microcracked medium described by the following bulk modulus and shear modulus:

$$\begin{cases} k_{\mathrm{eq}} = \dfrac{k}{1 + \frac{16}{9}\alpha\frac{1-\nu^2}{1-2\nu}} \cong k\left[1 - \dfrac{16}{9}\alpha\dfrac{1-\nu^2}{1-2\nu}\right] \\ \mu_{\mathrm{eq}} = \dfrac{\mu}{1 + \frac{32}{45}\alpha\frac{(1-\nu)(5-\nu)}{2-\nu}} \cong \mu\left[1 - \dfrac{32}{45}\alpha\dfrac{(1-\nu)(5-\nu)}{2-\nu}\right] \end{cases} \tag{49}$$

We may observe that the overall Hill parameters for an isotropic medium are given by: $k = k_{\mathrm{eq}} + (1/3)\mu_{\mathrm{eq}}$, $l = k_{\mathrm{eq}} - (2/3)\mu_{\mathrm{eq}}$, $n = (1/2)k_{\mathrm{eq}} + (2/3)\mu_{\mathrm{eq}}$ and $p = m = \mu_{\mathrm{eq}}$. In Eq. (49) we have also indicated the first order expansions because they will be used in the next section where some generalisations of these results will be described in order to consider higher values for the cracks density.

5. 3D isotropic distribution of cracks: generalisations

When the distribution of circular cracks is isotropic in a given spatial region, Eq. (49) furnishes the bulk modulus and the shear modulus of the overall system under the hypothesis of low density of cracks. These relations may be converted to similar ones describing the equivalent Young modulus and the effective Poisson ratio of the microcracked solid; to this aim we use Eq. (3) and we obtain the following results:

$$\begin{cases} E_{\mathrm{eq}} = \frac{E}{1+\frac{16}{45}\alpha\frac{(10-3\nu)(1-\nu^2)}{2-\nu}} \cong E\left[1 - \frac{16}{45}\alpha\frac{(10-3\nu)(1-\nu^2)}{2-\nu}\right] \\ \nu_{\mathrm{eq}} = \frac{\nu}{1+48\alpha\frac{(3-\nu)(1-\nu^2)}{16(1-\nu^2)\alpha+90-45\nu}} \cong \nu\left[1 - \frac{16}{15}\alpha\frac{(3-\nu)(1-\nu^2)}{2-\nu}\right] \end{cases} \tag{50}$$

In order to generalise the previous results to higher values of the cracks density we may adopt the differential scheme, exactly in the same way already used in the 2D case; so, a similar procedure allows us to obtain the following differential equations for the effective moduli of the material:

$$\begin{cases} \dfrac{\mathrm{d}E_{\mathrm{eq}}}{\mathrm{d}N} = -\dfrac{16}{45}\dfrac{a^3}{V}\dfrac{(10-3\nu_{\mathrm{eq}})(1-\nu_{\mathrm{eq}}^2)}{2-\nu_{\mathrm{eq}}}E_{\mathrm{eq}} \\ \dfrac{\mathrm{d}\nu_{\mathrm{eq}}}{\mathrm{d}N} = -\dfrac{16}{15}\dfrac{a^3}{V}\dfrac{(3-\nu_{\mathrm{eq}})(1-\nu_{\mathrm{eq}}^2)}{2-\nu_{\mathrm{eq}}}\nu_{\mathrm{eq}} \end{cases} \tag{51}$$

This differential problem may be solved obtaining implicit solutions:

$$\begin{cases} \left(\frac{\nu_{eq}}{\nu}\right)^{2/3}\left(\frac{1-\nu}{1-\nu_{eq}}\right)^{1/4}\left(\frac{1+\nu}{1+\nu_{eq}}\right)^{3/8}\left(\frac{3-\nu}{3-\nu_{eq}}\right)^{1/4} = e^{-\frac{16}{15}\alpha} \\ E_{eq} = E\left(\frac{\nu_{eq}}{\nu}\right)^{10/9}\left(\frac{3-\nu}{3-\nu_{eq}}\right)^{1/9} \end{cases} \tag{52}$$

Here, constants E and ν are elastic moduli of the matrix medium and constants E_{eq} and ν_{eq} are those of the microcracked solid. The parameter α is defined in Eq. (43). In this system (Eq. (52)), once the first irrational equation is (numerically) solved with respect to ν_{eq}, the Young modulus of the structure is directly given by the second expression. It may be interesting to observe that, as in the 2D case, the cracks density intervenes only by means of an exponential term, confirming the strong effects of the microcracking process over the mechanical properties of the solid. Results for the 3D case described in Eq. (52) have been represented in Fig. 4, together with the 2D case. It must be noticed that the parameter α for the 2D case and the 3D case has a different definition (see Eqs. (16) and (43)).

6. Conclusions

In this work we have analysed the effects of the orientational order/disorder of cracks in a homogeneous solid and we have described an explicit procedure that permits to obtain the mechanical behaviour of the microcracked medium. As additional result of this analysis we have found the correct definition of some order parameters in such a way to predict the macroscopic elastic properties as function of the state of microscopic order. In particular we have found that one order parameter is sufficient to describe the orientational distribution of slit-cracks in two-dimensional elasticity. On the other hand, the degree of ordering for circular cracks in three-dimensional elasticity is taken into account by means of two order parameters. These quantities are important to macroscopically characterise a given microcracked solid when the cracks density is fixed inside the materials. As results of great engineering significance we have found several relationships that furnish the stiffness tensor of the microcracked materials in terms of the microcracking features (in the case of low cracks density). Under the particular hypothesis of isotropic distribution of cracks we have obtained more accurate results that are correct for higher value of the cracks density. To do this we have used the iterative homogenisation method and the differential schemes.

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