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ON SOME EXPLICIT RESULTS FOR THE BALANCED GENERALIZED PÓLYA URN

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Abstract

This article describes an explicit approach to urn models of the balanced generalized Pólya type (with two types of balls). The treatment starts by obtaining the difference equations, describing the discrete time behavior of the expected value and of the variance of the selection probability for a given type of ball. The explicit solutions of such difference equations have been found in terms of gamma and psi (digamma) functions. This unified approach is useful in didactics in order to present a general method that leads to the final results without using complicated analytical tools. The more advanced mathematical procedure utilized is the solution of a first-order difference equation. All the theoretical results have been confirmed by a series of Monte Carlo simulations in order to clarify and better explain the behavior of the system.

Keywords: Pólya scheme; urn dynamics; difference equation

2000 Mathematics Subject Classification: Primary 37A50; 60F05

1. Introduction

The extended Pólya–Eggenberger models have been recently used for modeling several stochastic natural and technological systems ranging from computational biology to economic models or to advanced technology (for example, discrete channel description in communication theory and image processing).

An example in computational biology is given by the simulation of evolution of the genome structure: it has been verified that the combination of basic DNA sequence changes, described by urn models, can represent most of the natural DNA evolution events (i.e. deletion, insertion, point mutation, tandem repeats and transposition; see, for example, Zhou and Mishra (2004)).

The Pólya urn approach in economics considers increasing populations of entities in which, at each step in discrete time, one entity is probabilistically added according to an allocation function. This allows for modeling of complex markets and developing models in evolutionary economics (see Arthur (1994) and Arthur *et al.* (1963)).

Recently, the generalized urn schemes have been used by Alajaji and Fuja (1994) in modeling some discrete channels in communication theory, and have been applied by Banerjee *et al.* (1999) to create innovative algorithms for image segmentation and labeling.

The wide diffusion of the generalizations of the Pólya urn creates great interest in the analysis of the behavior of the general scheme, described below. A 2×2 matrix gives the replacement

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rules for the balanced generalized Pólya scheme:

$$A = \begin{pmatrix} \alpha & s-\alpha \\ s-\beta & \beta \end{pmatrix}.$$

The composition of the urn at the initial time is fixed and known (n white balls and m black balls, say). At a generic discrete time k, a ball in the urn is randomly chosen and its color is inspected (thus, the ball is drawn, looked at, and then placed back in the urn); if it is white then α white and $s-\alpha$ black balls are subsequently inserted, and if it is black then β black balls and $s-\beta$ white balls are inserted. This is represented by the 2×2 matrix A above. The balancing results from the following fact: independent of the color of the ball extracted at a given discrete time, s balls are added to the urn (the constant row sum is s). This means that the total number of balls in the urn is not a stochastic variable, but it is deterministically known. Obviously, the fraction of balls of a given color remains a stochastic variable. The probabilistic analysis of its behavior, in explicit closed form, is the main purpose of this paper. The generalized model, described by the matrix A, contains a series of schemes, which can be summed up as follows.

1. Drawing with replacement: $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. As is well known, the problem leads to the binomial probability law (for an example involving Bernoulli trials see, for example, Feller (1971)).

2. Drawing without replacement: $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The solution here involves the hypergeometric probability law (see Feller (1971)).

3. The Laplace *melancholic* model (see Laplace (1819)): $A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$. The replacement scheme describes the simple rule: if a ball is drawn, it is repainted black and returned to the urn no matter what its original color is.

4. The Ehrenfest–Ehrenfest model (see Ehrenfest and Ehrenfest (1907)): $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. This is related to a heat transfer model, which plays a crucial role in the discussion of the contradiction between irreversibility and ergodicity.

5. The Pólya–Eggenberger model (see Pólya and Eggenberger (1923)): $A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$. A ball is drawn at random and then replaced, together with α balls of the same color. It is a model of positive influence, known as the *contagion* model. It has been used in the epidemic modeling study of the spread of contagious diseases.

6. Adverse influence model: $A = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$. A ball is drawn at random and then replaced, together with α balls of the other color. This is also used in epidemiology and is called the *altruistic* model because choosing a white ball adds to the number of black balls in the urn and choosing a black ball adds to the number of white balls in the urn.

In general, the generalized model described by A is a problem with three parameters (α, β, and s) and two initial conditions (n white balls and m black balls).

The increasing number of applications of the generic urn model and its great importance from a theoretical point of view are the two main reasons for the consideration of this topic in many graduate and postgraduate courses. The aim of this paper is to present a mathematical method for analyzing the arbitrary balanced replacement scheme that is suitable for advanced courses on the theory of probability and statistical mathematics. The final results can be obtained with some long but quite elementary calculations. We have applied the methodology of the difference equations to both the average values and to the variance of the process. Moreover, the development of this topic in the present form can be useful to connect the world of random walks with simple and intuitive urn models.

2. Difference equations for the generalized urn model

To begin the analysis of the behavior of the generalized urn model we first consider an arbitrary nonbalanced replacement scheme described by the following 2×2 matrix:

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

From a probabilistic point of view, the stochastic dynamics of the system can be viewed as a two-dimensional random walk over a discrete grid defined on a plane by means of the two vectors $\boldsymbol{AB} = (a, b)$ and $\boldsymbol{AC} = (c, d)$, which represent the sides of a parallelogram (i.e. cell

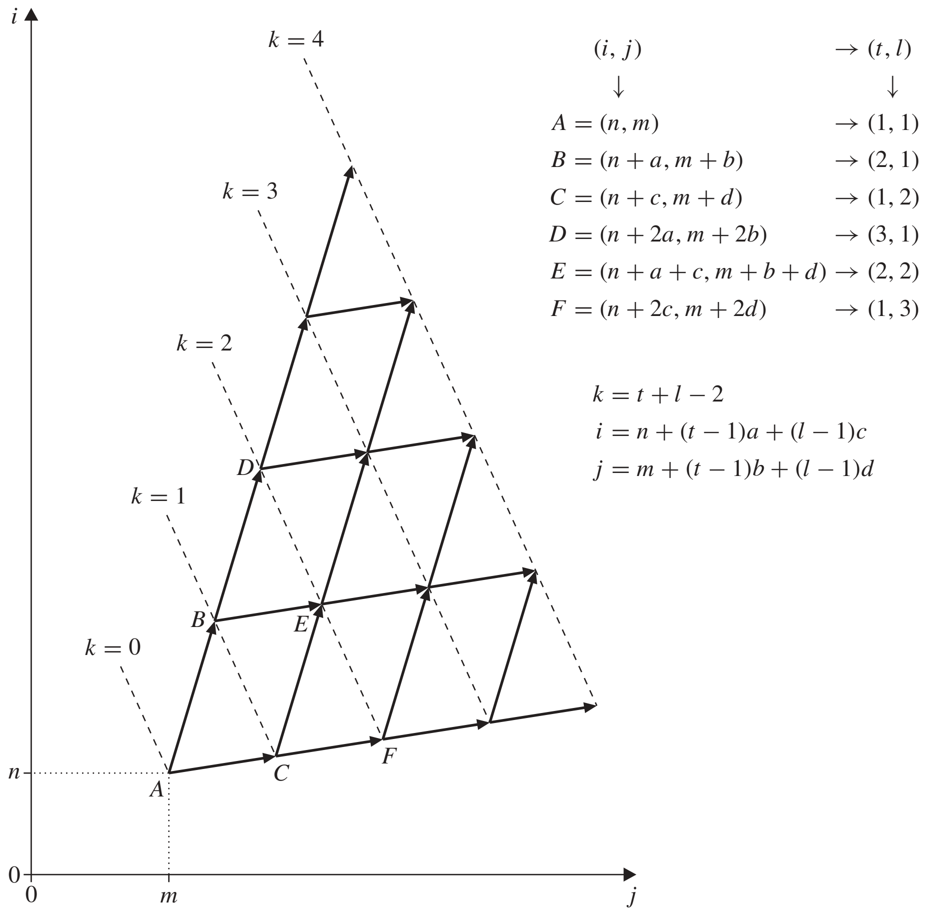

FIGURE 1: Two-dimensional random walk representation of the extended urn process. Each point of the plane (i, j) describes the occupational state of the urn (i white balls and j black balls). The two vectors $\boldsymbol{AB}$ and $\boldsymbol{AC}$ represent a geometrical interpretation of the matrix $\boldsymbol{A}$ regulating the stochastic replacement policy of balls.

of the grid). For a given point of the plane, the integer coordinates (i, j) represent the number of white and black balls inside the urn (see Figure 1 for details). Thus, at any point of the plane (i.e. for any state) an occupation probability $P_{i,j}$ is associated. This is the probability that the process passes from this point at some step. The initial condition is $P_{n,m} = 1$, if we begin with exactly n white balls and m black balls. The recursive relation over the grid is given by

$$P_{i,j} = P_{i-c,j-d} \frac{j-d}{i+j-c-d} + P_{i-a,j-b} \frac{i-a}{i+j-a-b}. \tag{1}$$

For computational reasons, instead of using the variables i and j, it is convenient to introduce the variables t and l, by means of the relations $i = n+(t-1)a+(l-1)c$ and $j = m+(t-1)b+(l-1)d$. By so doing, the enumeration of the vertices of the grid is very simple, as indicated in Figure 1. Moreover, it is useful to introduce another index, k, which counts the successive steps of the process; $k = 0$ at the beginning, $k = 1$ after the first drawing, $k = 2$ after the second drawing, and so on. It is easy to observe that $k = t + l - 2$. At step k, we have $k + 1$ possible states and, thus, $k + 1$ different occupation probabilities. When we consider a generic state identified by the point (i, j) we compute the selection probability for a white ball as $i/(i + j)$. By taking into account all the states belonging to the kth step, we may compute the mean value of the selection probabilities (weighted with the corresponding occupation probabilities) obtaining the average value of the selection probability p_k at the kth step. Similarly, the mean square value (which is always weighted with the corresponding occupation probabilities) of the differences between the selection probabilities and their average value represents the variance σ_k^2 of the selection probability at the kth step. We are interested in obtaining a recurrence relation for the variables p_k and σ_k^2.

To this purpose, we may define the probabilities as functions of the indices (t, l) (see Figure 1) by letting $Q_{t,l} = P_{n+(t-1)a+(l-1)c, m+(t-1)b+(l-1)d}$, so that (1) may be written in the following form:

$$\begin{aligned} Q_{t,l} = {} & Q_{t-1,l} \frac{n+(t-2)a+(l-1)c}{n+m+(t-2)(a+b)+(l-1)(c+d)} \\ & + Q_{t,l-1} \frac{m+(t-1)b+(l-2)d}{n+m+(t-1)(a+b)+(l-2)(c+d)}. \end{aligned} \tag{2}$$

Equation (2) is initialized with $Q_{1,1} = 1$. Recalling that the step index k is defined as $k = t+l-2$, we can explicitly define the expected value and the variance of the selection probability by means of the following weighted sums over the $k + 1$ states of the kth step:

$$\begin{aligned} p_k &= \sum_{t=1}^{k+1} Q_{t,k+2-t} \frac{n+(t-1)a+(k+1-t)c}{n+m+(t-1)(a+b)+(k+1-t)(c+d)}, \\ \sigma_k^2 &= \sum_{t=1}^{k+1} Q_{t,k+2-t} \left[\frac{n+(t-1)a+(k+1-t)c}{n+m+(t-1)(a+b)+(k+1-t)(c+d)} - p_k \right]^2. \end{aligned} \tag{3}$$

Combining (2) with (3) we note that it is possible to derive difference equations for p_k and σ_k^2 only if $a + b = c + d$, which means balancing of the replacement scheme. So, from now on, we let $a = \alpha$, $b = s - \alpha$, $c = s - \beta$, and $d = \beta$. So, letting $l = k+2-t$ in (2) and substituting this into (3), we may build up recursive exact relations for p_k and σ_k^2. After some long but straightforward algebraic computations we may write down these difference equations in their final form as

$$
\begin{aligned}
p_k &= \frac{(\alpha+\beta-s)+[n+m+(k-1)s]}{n+m+ks}p_{k-1}+\frac{s-\beta}{n+m+ks},\\
p_0 &= \frac{n}{n+m}
\end{aligned}
\tag{4}
$$

for the average value of the selection probability and

$$
\begin{aligned}
\sigma_k^2 &= \frac{[n+m+(k-1)s][2\alpha+2\beta+n+m+(k-3)s]}{(n+m+ks)^2}\sigma_{k-1}^2\\
&\quad+\frac{(\alpha+\beta-s)^2}{(n+m+ks)^2}p_{k-1}(1-p_{k-1}),\\
\sigma_0^2 &= 0
\end{aligned}
\tag{5}
$$

for its variance.

From a mathematical point of view these are nonhomogeneous first-order linear difference equations with nonconstant coefficients (depending on k). The main purpose of the paper is to derive some properties of the stochastic urn system by solving and analyzing (4) and (5).

3. Explicit solutions of the generalized urn model

Both the equations for the selection probability and the variance may be recast in the unified form

$$
x_k = a_k x_{k-1} + b_k, \tag{6}
$$

where a_k and b_k represent known sequences of real numbers and the first value x_0 is fixed and given. The general solution to (6), as known, is given by the expression

$$
x_k = \biggl(\prod_{j=1}^{k} a_j\biggr)x_0 + \sum_{i=1}^{k-1}\biggl(\prod_{j=i+1}^{k} a_j\biggr)b_i + b_k. \tag{7}
$$

An introduction to recurrence equations and their solutions and applications can be found in Goldberg (1986), Levy and Lessman (1992), and Agarwal (2000). We apply the methodology described by (7) to (4) and (5). We study the urn dynamics when $s \neq 0$, which is the more interesting case; if $s = 0$ the difference equations for p_k and σ_k^2 become linear with constant coefficients and the solutions are trivial (again, see Goldberg (1986), Levy and Lessman (1992), and Agarwal (2000)).

We start with the recursive relation (4) for the probability p_k. In this case, the second product appearing in (7) may be developed as follows (the first product is immediately obtained by letting $i = 0$):

$$
\begin{aligned}
\prod_{j=i+1}^{k} a_j &= \frac{(n+m+\alpha+\beta)/s+i-1}{(n+m)/s+i+1}\times\cdots\times\frac{(n+m+\alpha+\beta)/s+k-2}{(n+m)/s+k}\\
&= \frac{\Gamma((n+m+\alpha+\beta)/s+k-1)}{\Gamma((n+m+\alpha+\beta)/s+i-1)}\frac{\Gamma((n+m)/s+i+1)}{\Gamma((n+m)/s+k+1)}\\
&= \frac{\Gamma(p+k)}{\Gamma(p+i)}\frac{\Gamma(q+i)}{\Gamma(q+k)}.
\end{aligned}
$$

Here, we have defined $p = (n+m+\alpha+\beta-s)/s$ and $q = (n+m+s)/s$ and we have used the following property of the gamma function (which holds for any real y and for k and i integers):

$$(y+i)\cdots(y+k) = \frac{\Gamma(y+k+1)}{\Gamma(y+i)}.$$

Therefore, applying (7) to (4) furnishes, after some simple rearrangements, the following solution:

$$p_k = \frac{\Gamma(p+k)}{\Gamma(p)}\frac{\Gamma(q)}{\Gamma(q+k)}\frac{n}{n+m} + \frac{s-\beta}{s}\frac{\Gamma(p+k)}{\Gamma(q+k)}\sum_{i=1}^{k}\frac{\Gamma(q+i-1)}{\Gamma(p+i)}. \tag{8}$$

Now, the following additional result for the gamma function may be used to calculate the sum appearing in (8):

$$\sum_{i=1}^{k}\frac{\Gamma(a+i-1)}{\Gamma(b+i)} = \begin{cases} \dfrac{\Gamma(a+k)\Gamma(b)-\Gamma(a)\Gamma(b+k)}{(a-b)\Gamma(b)\Gamma(b+k)} & \text{if } a \neq b, \\ \Psi(a+k)-\Psi(a) & \text{if } a = b. \end{cases} \tag{9}$$

Here, the psi function is defined as $\Psi(x) = (\mathrm{d}/\mathrm{d}x)\ln\Gamma(x)$. Equation (9) can be easily proved by means of mathematical induction for $a \neq b$. For $a = b$ the psi function appears and the property represents the well-known functional relation

$$\Psi(x+n) = \Psi(x) + \sum_{k=0}^{n-1}\frac{1}{x+k}$$

(see, for example, Abramowitz and Stegun (1970)). By using the sum property (9) in (8) and identifying a with q and b with p, we may obtain the final solution for the selection probability of the white balls at the kth step as follows:

$$p_k = \begin{cases} \dfrac{s-\beta}{2s-\alpha-\beta} + \left(\dfrac{n}{n+m} - \dfrac{s-\beta}{2s-\alpha-\beta}\right)\dfrac{\Gamma(p+k)\Gamma(q)}{\Gamma(q+k)\Gamma(p)} & \text{if } 2s \neq \alpha+\beta, \\ \dfrac{n}{n+m} + \dfrac{\alpha-\beta}{\alpha+\beta}[\Psi(p+k)-\Psi(p)] & \text{if } 2s = \alpha+\beta. \end{cases} \tag{10}$$

Recalling the definitions of

$$p = \frac{n+m+\alpha+\beta-s}{s} \quad \text{and} \quad q = \frac{n+m+s}{s},$$

we observe that the condition $p = q$ is equivalent to $2s = \alpha+\beta$, as indicated in (10). This final relation for the dynamics of the selection probability of white balls is the first result of our analysis. We have postponed various comments about the limits of validity of this formula and consequences of it to Section 4. Here, we simply observe that the following property holds. If the matrix $\boldsymbol{A}$ has nonnegative entries then the asymptotic value of the probability (as $k \to \infty$) is $(s-\beta)/(2s-\alpha-\beta)$ if $2s \neq \alpha+\beta$ and $n/(n+m)$ if $s = \alpha = \beta$. (If $2s = \alpha+\beta$ then we may obtain nonnegative entries only if $s = \alpha = \beta$ and therefore the selection probability remains constant at the value $n/(n+m)$.)

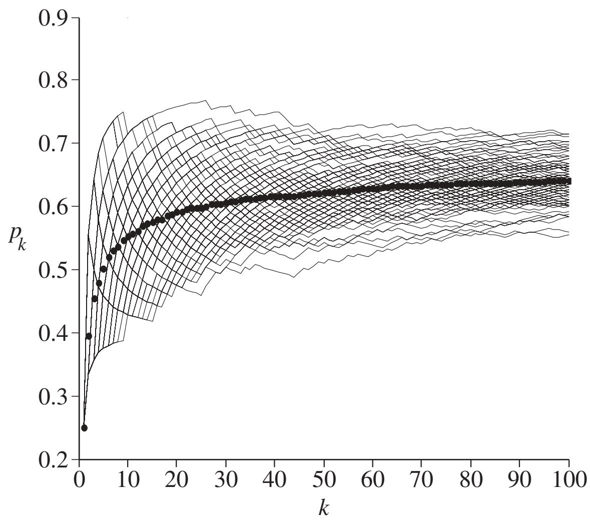

FIGURE 2: Results of the Monte Carlo simulation performed on the urn described by $n = 1$, $m = 3$, $s = 5$, $\alpha = 4$, and $\beta = 3$. We have generated 100 instances of the process, each of them with 100 extractions. The solid lines represent the selection probability of all the instances and the dotted line represents the corresponding average value.

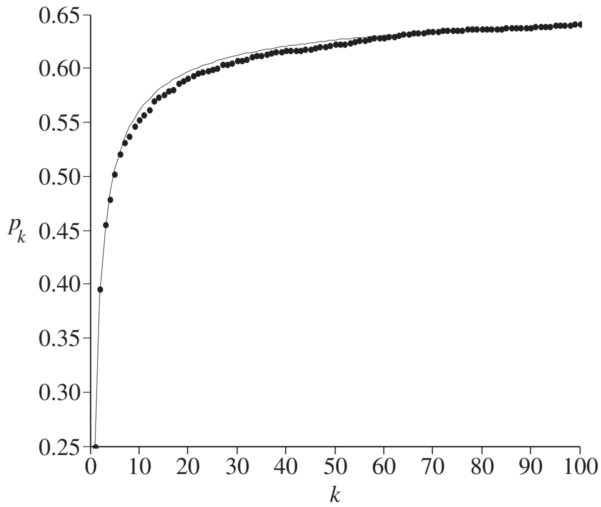

FIGURE 3: Results for the urn described by $n = 1$, $m = 3$, $s = 5$, $\alpha = 4$, and $\beta = 3$. A comparison between the average value of the selection probability obtained with the Monte Carlo method (dotted line) and the theoretical prediction given by (10) (solid line) is shown.

We now present some numerical simulations that describe such results. We consider the first case with $n = 1$, $m = 3$, $s = 5$, $\alpha = 4$, and $\beta = 3$. We simulate the urn process by means of the Monte Carlo technique performing 100 instances of the process, each of them consisting of 100 steps (i.e. extractions). All the instances start from the same initial conditions and the drawing of a ball is mimed by means of a uniform random number generator. The software code has been developed using MATLAB®. All the instances evolve independently following different paths on the grid shown in Figure 1. Therefore, all such independent instances have been used to obtain the requested average values, which are the selection probability and the variance. The results concerning the variance will be discussed later on in this section. In Figure 2, we show the dynamics of the selection probability; the solid lines represent the actual selection probabilities related to all the instances and the dotted line corresponds to the average value computed over all these instances. It is interesting to observe that the structure with many rhombuses generated by the solid lines in Figure 2 corresponds to the different paths of the random walk shown in Figure 1. Moreover, in Figure 3 we draw a comparison between the average value of the selection probability obtained with the Monte Carlo simulations and the theoretical prediction given by (10). We can observe the good agreement between the numerical and theoretical achievements and the convergence of the selection probability to the value

$$\frac{s-\beta}{2s-\alpha-\beta}=\frac{2}{3}.$$

Now, to complete the analytical procedure, we apply (7) to solve the variance relation (5); the related product can be written as follows (by taking the generic term a_j from (5)):

$$\begin{aligned}\prod_{j=i+1}^{k} a_j &= \frac{((n+m)/s+i)((n+m+2\alpha+2\beta)/s+i-2)}{((n+m)/s+i+1)^2}\\ &\quad\times\cdots\times\frac{((n+m)/s+k-1)((n+m+2\alpha+2\beta)/s+k-3)}{((n+m)/s+k)^2}\\ &=\frac{\Gamma((n+m+2\alpha+2\beta)/s+k-2)}{\Gamma((n+m+2\alpha+2\beta)/s+i-2)}\frac{\Gamma((n+m)/s+k)}{\Gamma((n+m)/s+i)}\\ &\quad\times\frac{\Gamma^2((n+m)/s+i+1)}{\Gamma^2((n+m)/s+k+1)}\\ &=\frac{\Gamma(2p-q+k+1)}{\Gamma(2p-q+i+1)}\frac{\Gamma(q+i)}{\Gamma(q+k)}\frac{q+i-1}{q+k-1}.\end{aligned}\tag{11}$$

Here, we have maintained the previous definitions of p and q. Substituting (11) into (7) with $x_0=0$, i.e. $\sigma_0^2=0$, we obtain the following first form for the variance (which is correct for any value of the characteristic parameters):

$$\sigma_k^2=\frac{\Gamma(2p-q+k+1)}{\Gamma(q+k)}\frac{(\alpha+\beta-s)^2}{s(n+m+ks)}\sum_{i=1}^{k}\frac{\Gamma(q+i-1)}{\Gamma(2p-q+i+1)}p_{i-1}(1-p_{i-1})\tag{12}$$

(of course $s\neq 0$ as before). To further simplify this expression, we should substitute (10) into (12) in order to perform the sum on the right-hand side. This calculation is very long and, thus, we give here the main results, avoiding some intermediate details. The procedure is very similar in spirit to the previous one.

If $p\neq q$ or, equivalently, $2s\neq\alpha+\beta$, we take the first formula in (10), substitute it into (12), and (again using the addition rule for the gamma function (9)) we obtain two different results: the first is valid if $q-p\neq\frac{1}{2}$ and the other one if $q-p=\frac{1}{2}$.

Summing up, if $p \neq q$ and $q - p \neq \frac{1}{2}$ (i.e. $2s \neq \alpha + \beta$ and $3s \neq 2\alpha + 2\beta$) the variance is given by

$$\begin{aligned}\sigma_k^2 = {} & \frac{(\alpha+\beta-s)^2(s-\alpha)(s-\beta)}{(\alpha+\beta-2s)^2(3s-2\alpha-2\beta)(n+m+ks)}\left[1-\frac{\Gamma(2p-q+k+1)\Gamma(q)}{\Gamma(q+k)\Gamma(2p-q+1)}\right] \\ & + \frac{(s-\alpha-\beta)(\beta-\alpha)[n(s-\alpha)-m(s-\beta)]}{(\alpha+\beta-2s)^2(n+m)(n+m+ks)} \\ & \times \left[\frac{\Gamma(p+k)\Gamma(q)}{\Gamma(q+k)\Gamma(p)} - \frac{\Gamma(2p-q+k+1)\Gamma(q)}{\Gamma(q+k)\Gamma(2p-q+1)}\right] \\ & + \frac{[n(s-\alpha)-m(s-\beta)]^2}{(\alpha+\beta-2s)^2(n+m)^2} \\ & \times \left[\frac{(n+m)}{(n+m+ks)}\frac{\Gamma(2p-q+k+1)\Gamma(q)}{\Gamma(q+k)\Gamma(2p-q+1)} - \frac{\Gamma^2(p+k)\Gamma^2(q)}{\Gamma^2(q+k)\Gamma^2(p)}\right]. \end{aligned} \tag{13}$$

As an example of an application of this relation, we may use the urn scheme simulated in the first part of this section. Therefore, we consider the scheme characterized by the values $n = 1$, $m = 3$, $s = 5$, $\alpha = 4$, and $\beta = 3$ (the relations $2s \neq \alpha + \beta$ and $3s \neq 2\alpha + 2\beta$ are clearly satisfied). As before, we draw a comparison between the variance dynamics obtained with the Monte Carlo method (100 instances with 100 steps) with the exact result given in (13). The results are shown in Figure 4, where a good agreement can be observed. Moreover, we note that in such a case, the variance σ_k^2 approaches zero for large values of k.

However, if $p \neq q$ (i.e. $2s \neq \alpha + \beta$) and $q - p = \frac{1}{2}$ (i.e. $3s = 2\alpha + 2\beta$), we have $s = \frac{2}{3}(\alpha + \beta)$ and the replacement matrix assumes the form

$$\boldsymbol{A} = \begin{pmatrix} \alpha & \dfrac{2\beta-\alpha}{3} \\ \dfrac{2\alpha-\beta}{3} & \beta \end{pmatrix}.$$

In this case, we may let $(2\beta - \alpha)/3 = N$ and $(2\alpha - \beta)/3 = M$, so that we obtain

$$\boldsymbol{A} = \begin{pmatrix} 2M+N & N \\ M & 2N+M \end{pmatrix},$$

where N and M are integers. This special replacement policy leads to a specific relationship for its variance dynamics. For computational convenience, it is better to analyze this urn scheme in terms of N and M instead of α and β. With these assumptions, the variable p assumes the value $p = (n + m + N + M)/(2N + 2M)$ (and $q = p + \frac{1}{2}$). When the matrix $\boldsymbol{A}$ has the form specified above, the variance is given by the following special expression, written in terms of N and M:

$$\begin{aligned}\sigma_k^2 = {} & \frac{1}{2}\frac{MN}{(N+M)(n+m+2kN+2kM)}\left[\Psi\left(\frac{1}{2}+p+k\right)-\Psi\left(\frac{1}{2}+p\right)\right] \\ & + \frac{(N-M)(nN-mM)}{(N+M)(n+m)(n+m+2kN+2kM)}\left[1-\frac{\Gamma(\frac{1}{2}+p)\Gamma(p+k)}{\Gamma(\frac{1}{2}+p+k)\Gamma(p)}\right] \\ & + \frac{(nN-mM)^2}{(N+M)^2(n+m)^2}\left[\frac{n+m}{n+m+2kN+2kM} - \frac{\Gamma^2(\frac{1}{2}+p)\Gamma^2(p+k)}{\Gamma^2(\frac{1}{2}+p+k)\Gamma^2(p)}\right]. \end{aligned} \tag{14}$$

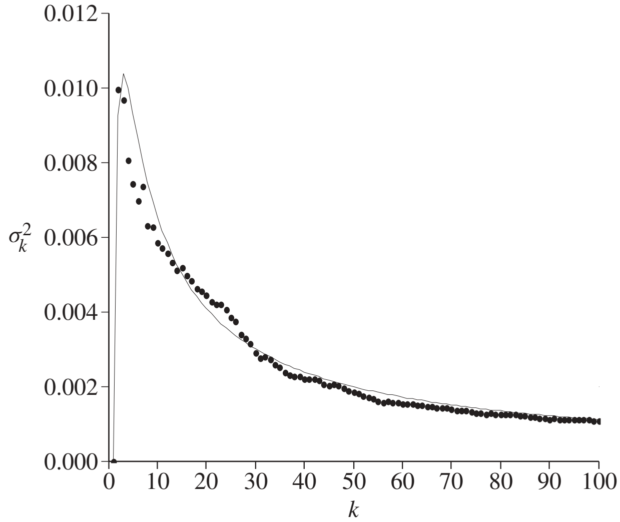

FIGURE 4: Comparison between the average value of the variance obtained with the Monte Carlo method (dotted line) and the theoretical prediction given by (13) (solid line) is shown. The results correspond to the urn model described by $n = 1$, $m = 3$, $s = 5$, $\alpha = 4$, and $\beta = 3$.

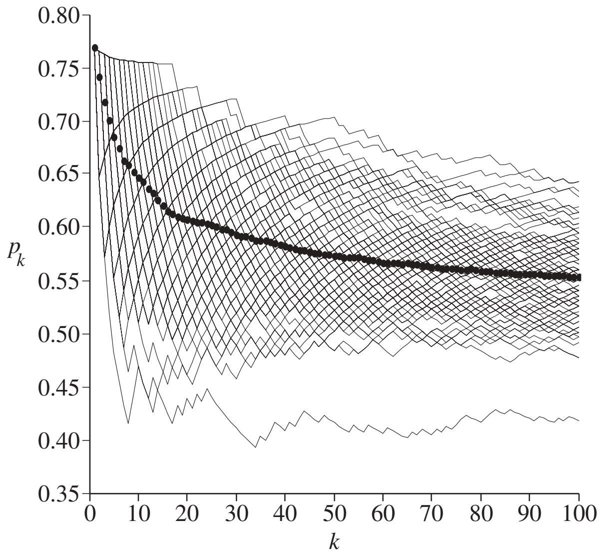

FIGURE 5: Dynamics of the selection probability for the case with $\alpha = 3$, $\beta = 3$, $s = 4$, $n = 10$, and $m = 3$ (100 Monte Carlo instances with 100 steps). The solid lines represent the probability evolution of each instance and the dotted line corresponds to its average value.

We now describe a complete set of simulations pertinent to such a case. We adopt the following parameters: $\alpha = 3$, $\beta = 3$, $s = 4$, $n = 10$, and $m = 3$. The relations $2s \neq \alpha + \beta$ and $3s = 2\alpha + 2\beta$ are clearly satisfied, and therefore we define the alternative quantities $N = 1$ and $M = 1$. The results of the Monte Carlo simulations (100 instances with 100 steps) are shown in Figures 5, 6, and 7. In Figure 5 the dynamics of the selection probability is shown; as before, the

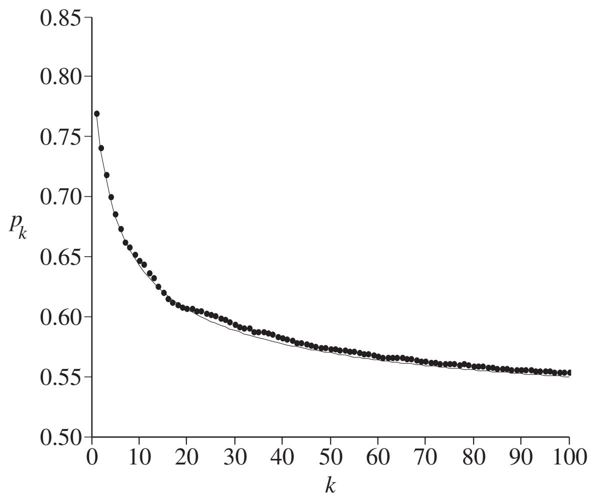

FIGURE 6: Comparison between the average values of the selection probability obtained with the Monte Carlo simulations (dotted line) and the theoretical prediction given by (10) (solid line).

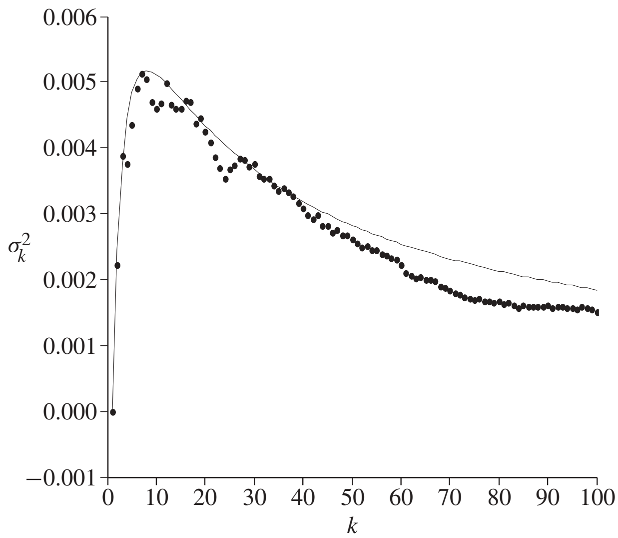

FIGURE 7: Comparison between the variance numerically obtained with Monte Carlo simulations (dotted line) and that calculated with (14) (solid line) for the urn with $\alpha = 3$, $\beta = 3$, $s = 4$, $n = 10$, and $m = 3$ ($N = 1$ and $M = 1$).

solid lines represent the probability evolution of each instance and the dotted lines correspond to its average value. In Figure 6 a comparison between the average value of the selection probability obtained with the Monte Carlo simulations and the theoretical prediction given by (10) is drawn. We can observe the slow convergence to the value $(s - \beta)/(2s - \alpha - \beta) = \frac{1}{2}$. Finally, in Figure 7 we show a comparison between the variance numerically obtained (using the Monte Carlo method) and that calculated with (14). Also in this case we note the convergence to the value $\sigma_k^2 = 0$ for large values of k.

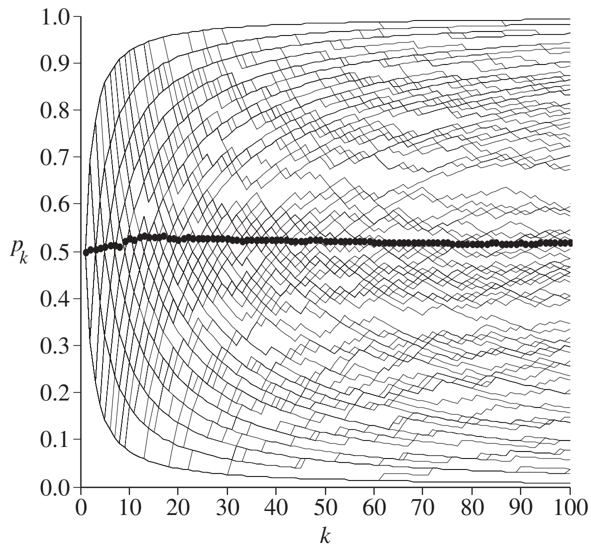

FIGURE 8: Evolution of the selection probability for a pure contagion urn model described by $\alpha = 5$ and $n = m = 4$. The theoretical results (solid lines) can be compared with the Monte Carlo simulations (dotted line).

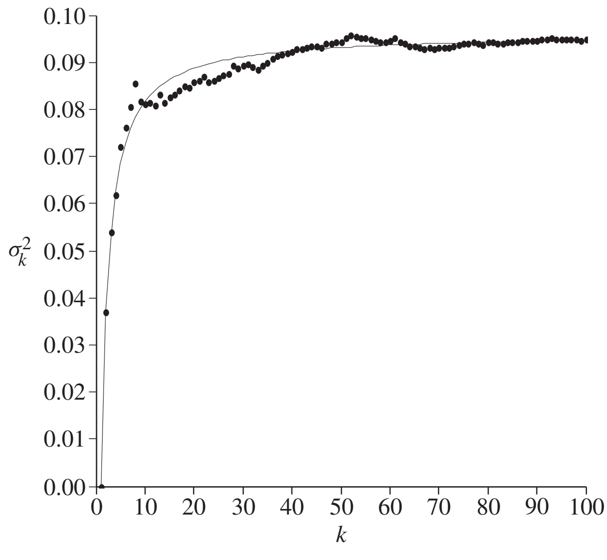

FIGURE 9: Evolution of the variance for a pure contagion urn model described by $\alpha = 5$ and $n = m = 4$. The theoretical results (solid line) can be compared with the Monte Carlo simulations (dotted line).

The last case deals with the assumption $p = q$ or, equivalently, $2s = \alpha + \beta$. In this case we should substitute the second formula of (10) into (12); unfortunately, the sum in (12) cannot be performed in closed form because it contains a complicated combination of gamma and psi functions. We may analyze this case only under the additional hypothesis $s = \alpha = \beta$, which corresponds to the contagion model. In this simple case the selection probability remains constant at the value $p_k = n/(n+m)$ (for any k) and we may write down the explicit solution for the variance as follows:

$$\sigma_k^2 = \frac{\alpha^2 nmk}{(n+m)^2(n+m+\alpha)(n+m+k\alpha)}. \tag{15}$$

The variance for large values of k approaches the value $\sigma_k^2 = \alpha nm/(n+m)^2(n+m+\alpha)$, which corresponds to the variance of the limiting density probability given by the beta distribution (this is the famous result of Pólya and Eggenberger (1923)). We performed a series of simulations with the values $s = \alpha = \beta = 5$ and $n = m = 4$. The selection probability should remain constant at the value $\frac{1}{2}$ and the variance should converge towards the value $\frac{5}{52} \approx 0.0961$. In Figure 8 we show the dynamics of the selection probability and in Figure 9 we show the variance. This is the sole case where the variance approaches a finite value (corresponding to the beta distribution) for large values of k.

As another example of an application of the previous theory, we may obtain the specific solutions for the adverse influence model (altruistic model) described by the replacement matrix $\boldsymbol{A} = \binom{0\ 1}{1\ 0}$, corresponding to $s = 1$ and $\alpha = \beta = 0$. A ball is drawn at random and then replaced, together with a ball of the other color. Equations (10) and (13) can be simplified to obtain

$$p_k = \frac{2k(n+m+1) + 2n(n+m-1) + k(k-3)}{2(n+m+k)(n+m+k-1)} \tag{16}$$

and

$$\begin{aligned}\sigma_k^2 = {} & \frac{1}{12}\frac{1}{n+m+k} - \frac{1}{12}\frac{(n+m-2)(n+m-1)(n+m)}{(n+m+k)^2(n+m+k-1)(n+m+k-2)} \\ & - \frac{1}{4}\frac{(n-m)^2(n+m-1)^2}{(n+m+k)^2(n+m+k-1)^2} \\ & + \frac{1}{4}\frac{(n-m)^2(n+m-1)(n+m-2)}{(n+m+k)^2(n+m+k-1)(n+m+k-2)}. \end{aligned} \tag{17}$$

In the limiting case of $k \to \infty$, we find that $p_k \to \frac{1}{2}$ (with order two) and $\sigma_k^2 \to 0$ (with order one). In Figure 10 the behavior of the selection probability and of the variance is shown drawing a comparison between (16) and (17) and some computer simulations. A general discussion about the limiting case of $k \to \infty$ follows in Section 4.

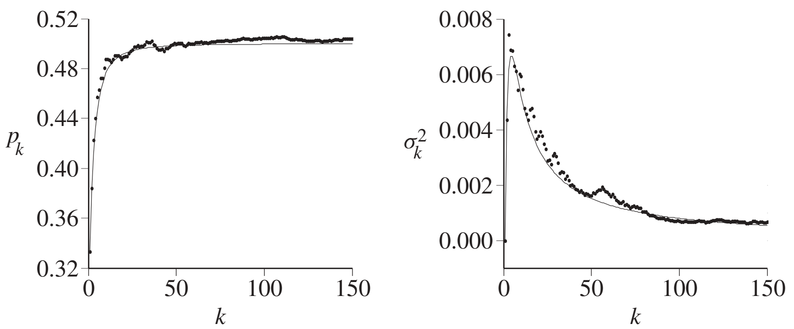

FIGURE 10: Evolution of the selection probability and of the variance for a pure altruistic urn model described by $\boldsymbol{A} = \binom{0\ 1}{1\ 0}$. Results obtained by (16) and (17) (solid lines) can be compared with the Monte Carlo simulations (dotted lines).

4. Discussion and conclusions

It must be emphasized that all the solutions given in Section 3 apply only with some limitations on the index variable k, which counts on the successive steps of the stochastic process. When all the entries of the replacement matrix are nonnegative, we may consider the problem with an infinite succession of steps, because balls are never removed from the urn and, therefore, the process will never stop. On the contrary, when one or more entries of the replacement matrix are negative, there is a probability of removing some balls from the urn at each step and consequently the process may reach an end when a type of ball is consumed. In other words, when some entries are negative in $\boldsymbol{A}$, the grid shown in Figure 1, which contains all the possible random walks, will be crossing the frontiers of the first quadrant. This happens for a given value of k referred to as k_{end}; all the final formulas of Section 3 are no longer valid for k greater than k_{end}.

Now, we are interested in the asymptotic behavior of the process, when all the entries of $\boldsymbol{A}$ are nonnegative. There are two cases, which exhibit different dynamics. If $\alpha \geq 0, \beta \geq 0, s-\beta \geq 0$, $s-\alpha \geq 0$, and $2s-\alpha-\beta > 0$, then all the entries are nonnegative and at most one entry on the secondary diagonal is strictly positive (describing some adverse influence or altruistic behavior). The sole other possibility to have all the entries nonnegative is $\alpha = \beta = s > 0$, which corresponds to the classical Pólya–Eggenberger contagious urn scheme. The former case is described by the following property.

If $\alpha \geq 0$, $\beta \geq 0$, $s-\beta \geq 0$, $s-\alpha \geq 0$, and $2s-\alpha-\beta > 0$, then the following limits are fulfilled:

$$\lim_{k\to\infty} p_k = \frac{s-\beta}{2s-\alpha-\beta} \quad \text{and} \quad \lim_{k\to\infty} \sigma_k^2 = 0, \tag{18}$$

and the following asymptotic properties hold:

$$\begin{aligned} p_k - \frac{s-\beta}{2s-\alpha-\beta} &\overset{k\to\infty}{\sim} \frac{C_1}{k^{(2s-\alpha-\beta)/s}}, \\ \sigma_k^2 &\overset{k\to\infty}{\sim} \begin{cases} \dfrac{C_2}{k^{2(2s-\alpha-\beta)/s}} & \text{if } 0 < \dfrac{2s-\alpha-\beta}{s} < \dfrac{1}{2}, \\ \dfrac{C_3 \ln k}{k} & \text{if } \dfrac{2s-\alpha-\beta}{s} = \dfrac{1}{2}, \\ \dfrac{C_4}{k} & \text{if } \dfrac{2s-\alpha-\beta}{s} > \dfrac{1}{2}, \end{cases} \end{aligned} \tag{19}$$

where the C_ks are finite constants. The three different velocities of convergence of the variance are shown in Figure 11 where the following values are taken into consideration. The initial conditions $n = 2$ and $m = 4$ have been fixed for the three cases studied. In the first case we assumed that $s = 5$, $\alpha = 4$, and $\beta = 3$ (triangles in Figure 11). This corresponds to $(2s-\alpha-\beta)/s = \frac{3}{5} > \frac{1}{2}$, i.e. to the faster convergence. The second case considers $s = 4$, $\alpha = 3$, and $\beta = 3$, and thus $(2s-\alpha-\beta)/s = \frac{1}{2}$ (circles in Figure 11). Finally, the third case with slower convergence assumes the values $s = 3$, $\alpha = 2$, and $\beta = 3$, from which we obtain $(2s-\alpha-\beta)/s = \frac{1}{3} < \frac{1}{2}$ (dots in Figure 11). For each case we have represented the theoretical prediction for the variance and the results of the Monte Carlo simulations.

In the latter case ($\alpha = \beta = s > 0$, i.e. the Pólya urn scheme), the average value of the selection probability remains constant during the whole process (this is evident by observing (10) with $\alpha = \beta$) at the value $n/(n+m)$; its variance converges to that of the beta distribution as suggested by the dynamics given by (15) (with rate of convergence of order one).

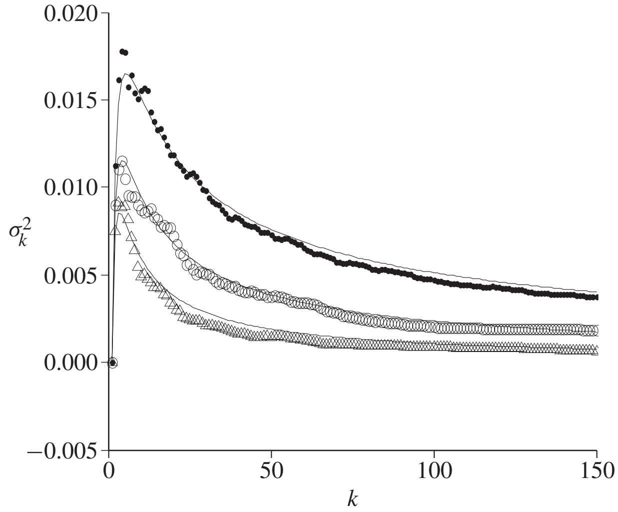

FIGURE 11: Comparison among the different convergence velocities of the variance ($n = 2$ and $m = 4$ have been fixed for the three cases). First case: $s = 5$, $\alpha = 4$, and $\beta = 3$ (triangles). Second case: $s = 4$, $\alpha = 3$, and $\beta = 3$ (circles). Third case: $s = 3$, $\alpha = 2$, and $\beta = 3$ (dots).

The proofs of (18) and (19) follow. These are based on the following properties:

(a) $\Gamma(n + a)/\Gamma(n + b) \sim n^{a-b}$,

(b) $\Psi(n) \sim \ln n$ (n approaching infinity)

(see, for example, Abramovitz and Stegun (1970)). Formulas given by (18) can be derived directly from (10), (13), and (14) performing the limits and assuming that $\alpha \geq 0$, $\beta \geq 0$, $s - \beta \geq 0$, $s - \alpha \geq 0$, and $2s - \alpha - \beta > 0$. Similarly, the asymptotic relation for p_k (the first in (19)) immediately follows from (10) and property (a).

To verify the asymptotic property for the variance we firstly assume that $\alpha \geq 0$, $\beta \geq 0$, $s-\beta \geq 0, s-\alpha \geq 0, 2s-\alpha-\beta > 0$, and $(2s - \alpha - \beta)/s \neq \frac{1}{2}$ (i.e. $q - p \neq \frac{1}{2}$ or $3s \neq 2\alpha+2\beta$); in this case (13) applies and it contains the following results at infinity (using property (a)):

$$\frac{1}{k}, \qquad \frac{1}{k^{2x}}, \qquad \frac{1}{k^{x+1}}$$

(where $x = q - p = (2s - \alpha - \beta)/s$). It follows that the leading term is $1/k^{2x}$ if $0 < x < \frac{1}{2}$ and $1/k$ if $x > \frac{1}{2}$.

Finally, if we assume that $\alpha \geq 0$, $\beta \geq 0$, $s - \beta \geq 0$, $s - \alpha \geq 0$, $2s - \alpha - \beta > 0$, and $(2s - \alpha - \beta)/s = \frac{1}{2}$ (i.e. $x = q - p = \frac{1}{2}$ or $3s = 2\alpha + 2\beta$), then (14) holds and it exhibits the following results at infinity (using properties (a) and (b)):

$$\frac{1}{k}, \qquad \frac{\ln k}{k}, \qquad \frac{1}{k^{3/2}}.$$

Therefore, the leading term is $\ln k/k$ if $x = \frac{1}{2}$. This completes the verification.

We now make some concluding remarks. As previously shown, when we are dealing with a pure contagious scheme, the selection probability for a given type of ball remains constant during the process evolution, while the fraction of balls reaches the beta distribution (see Pólya and Eggenberger (1923)). As soon as we add an altruistic contribution, the probability evolves in discrete time converging to the fixed value $(s-\beta)/(2s-\alpha-\beta)$ (independently of the contagious coefficients α and β). Moreover, the variance converges to zero (with rate of convergence given above), assuring that any instance of the process converges deterministically to the value of probability given by $(s-\beta)/(2s-\alpha-\beta)$. In other words, the altruistic behavior of the urn process is dominant over the contagious one.

Finally, we want to point out that the analysis of the urn systems has been performed in many introductory texts on probability theory, for example in Feller (1971) and Johnson and Kotz (1977). Thus, our approach is not new in principle but has been conducted analytically for the generalized Pólya urn. So, it unifies all the possible cases, typically analyzed separately, and furnishes closed form solutions to the difference equations describing the dynamics of average values and variances. The approach of solving the general difference equations in closed form could be useful for alternative didactic approaches. In particular, the explicit expressions for the variance behavior may be useful in several applications. The knowledge of the explicit solutions is also useful to obtain the asymptotic behavior of the system with different values of the parameters that define the urn scheme under consideration.

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