Supplemental Material Section of the manuscript:

Analytical procedure for determining the linear and nonlinear effective properties of the elastic composite cylinder

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In this Supplemental Material Section we present a Maple™ code able to determine the linear and nonlinear effective elastic moduli of a nonlinearly elastic composite cylinder. The code is based on the solutions given in Eqs.(47)-(50) of the main text where the coefficients have been renamed as follows for simplicity: d_{-3}=**M**, d_{-1}=**H**, d_1=**G**, c_{-1}=**F**, c_1=**N**, c_3=**Q**, b_1=**B**, a_1=**A** and a_3=**W**. By using the Kolossov-Muskhelishivili equations (see Eq.(25) of the main text) we have successively determined the distribution of displacement within the heterogeneous structure (variables **ux1, uy1, ux2, uy2** in the code). Therefore, we have straightforwardly calculated the strain components in the two phases (variables **exx1, eyy1, exy1, exx2, eyy2, exy2** in the code). In the following part of the code the terms appearing in Eqs.(66) and (67) have been implemented and all the integrals have been calculated analytically by means of the standard use of the cylindrical coordinates (see Table I for the names of variables in the Maple code). Finally, the linear **mueff, keff** and nonlinear **feff, eeff** effective elastic moduli have been implemented through Eqs.(68)-(71). The results are exact for any volume fraction **p** (over the range 0..1) of the core (inhomogeneity of radius **R**) in the external shell (matrix of radius **Rt**). To conclude, we remark that the code could be also used for verifying the second order expansions given in Eqs.(72)-(75), valid for nonlinear tubes or nanotubes and the first order expansion given in Eqs.(88)-(89) with all the coefficients reported in Appendix A.

Table I: Integrals appearing in Eqs.(66) and (67) and corresponding names in The maple code. The complete expression of each integral can be found within the code.

Integrals	Variable name
$\int_{\Omega_1} \left\{ \mathrm{Tr}[\hat{\epsilon}^L(\vec{x})] \right\}^2 d\vec{x}$	**intetre1**
$\int_{\Omega_2} \left\{ \mathrm{Tr}[\hat{\epsilon}^L(\vec{x})] \right\}^2 d\vec{x}$	**intetre2**
$\int_{\Omega_1} \mathrm{Det}[\hat{\epsilon}^L(\vec{x})] d\vec{x}$	**intedete1**
$\int_{\Omega_2} \mathrm{Det}[\hat{\epsilon}^L(\vec{x})] d\vec{x}$	**intedete2**
$\int_{\Omega_1} \mathrm{Tr}^3 \left[\hat{\epsilon}^L(\vec{x})\right] d\vec{x}$	**intetre1cubo**
$\int_{\Omega_2} \mathrm{Tr}^3 \left[\hat{\epsilon}^L(\vec{x})\right] d\vec{x}$	**intetre2cubo**
$\int_{\Omega_1} \mathrm{Tr}\left[\hat{\epsilon}^L(\vec{x})\right] \mathrm{Det}[\hat{\epsilon}^L(\vec{x})] d\vec{x}$	**intetrdet1**
$\int_{\Omega_2} \mathrm{Tr}\left[\hat{\epsilon}^L(\vec{x})\right] \mathrm{Det}[\hat{\epsilon}^L(\vec{x})] d\vec{x}$	**intetrdet2**

```
> chi1:=1+2*mu1/k1;chi2:=1+2*mu2/k2;
```

$$\chi 1 := 1 + \frac{2\,\mu 1}{k1}$$

$$\chi 2 := 1 + \frac{2\,\mu 2}{k2}$$

```
> phi1:=N*z+F/z+Q*z^3;
```

$$\phi 1 := N\,z + \frac{F}{z} + Q\,z^3$$

```
> psi1:=G*z+H/z+M/z^3;
```

$$\psi 1 := G\,z + \frac{H}{z} + \frac{M}{z^3}$$

```
> N:=Nr:F:=Fr+I*Fi:G:=Gr+I*Gi:H:=Hr:Q:=Qr+I*Qi:M:=Mr+I*Mi:
> phi2:=A*z+W*z^3;
> psi2:=B*z;
```

$$\phi 2 := A\,z + \mathrm{W}\,z^3$$

$$\psi 2 := B\,z$$

```
> B:=Br+I*Bi:A:=Ar:W:=Wr+I*Wi:
> phi11:=diff(phi1,z):
> phi12:=diff(phi11,z):
> phi21:=diff(phi2,z):
> phi22:=diff(phi21,z):
> psi11:=diff(psi1,z):
> psi21:=diff(psi2,z):
> z:=x+I*y;
>
```

$$z := x + y\,I$$

```
>
> dep1:=((1/2/mu1)*(chi1*phi1-z*conjugate(phi11)-conjugate(psi1)))
  :
> dep2:=((1/2/mu2)*(chi2*phi2-z*conjugate(phi21)-conjugate(psi2)))
  :
> ux1:=Re(dep1) assuming
  (x,real,y,real,mu1,real,mu2,real,k1,real,k2,real,Nr,real,Fr,real
  ,Fi,real,Gr,real,Gi,real,Hr,real,Qr,real,Qi,real,Mr,real,Mi,real
  ,Br,real,Bi,real,Ar,real,Wr,real,Wi,real):
> uy1:=Im(dep1) assuming
  (x,real,y,real,mu1,real,mu2,real,k1,real,k2,real,Nr,real,Fr,real
  ,Fi,real,Gr,real,Gi,real,Hr,real,Qr,real,Qi,real,Mr,real,Mi,real
  ,Br,real,Bi,real,Ar,real,Wr,real,Wi,real):
> ux2:=Re(dep2) assuming
  (x,real,y,real,mu1,real,mu2,real,k1,real,k2,real,Nr,real,Fr,real
  ,Fi,real,Gr,real,Gi,real,Hr,real,Qr,real,Qi,real,Mr,real,Mi,real
  ,Br,real,Bi,real,Ar,real,Wr,real,Wi,real):
> uy2:=Im(dep2) assuming
```

```
(x,real,y,real,mu1,real,mu2,real,k1,real,k2,real,Nr,real,Fr,real
,Fi,real,Gr,real,Gi,real,Hr,real,Qr,real,Qi,real,Mr,real,Mi,real
,Br,real,Bi,real,Ar,real,Wr,real,Wi,real):
> exx1:=simplify(diff(ux1,x));
```

$$exx1 := -(8\,\mu1\,Fi\,y^3\,x^3 + 4\,\mu1\,Fi\,y^5\,x - 12\,\mu1\,Qr\,x^6\,y^4 + 12\,\mu1\,Qr\,x^4\,y^6 + 18\,\mu1\,Qr\,x^2\,y^8 - Hr\,x^4\,k1\,y^2 + Hr\,x^2\,k1\,y^4 + 4\,Gr\,x^6\,k1\,y^2 + 6\,Gr\,x^4\,k1\,y^4 + 18\,k1\,Mr\,y^2\,x^2 - 12\,k1\,Mi\,y\,x^3 + 12\,k1\,Mi\,y^3\,x - 10\,k1\,Fr\,y^4\,x^2 + 8\,k1\,Fi\,y\,x^5 - 8\,k1\,Fi\,y^5\,x + 12\,\mu1\,Qi\,x^9\,y + 48\,\mu1\,Qi\,x^7\,y^3 + 72\,\mu1\,Qi\,x^5\,y^5 + 48\,\mu1\,Qi\,x^3\,y^7 + 4\,Gr\,x^2\,k1\,y^6 - 8\,\mu1\,Nr\,x^6\,y^2 - 12\,\mu1\,Nr\,x^4\,y^4 - 8\,\mu1\,Nr\,x^2\,y^6 + 2\,\mu1\,Fr\,x^4\,y^2 - 2\,\mu1\,Fr\,x^2\,y^4 + 4\,\mu1\,Fi\,y\,x^5 + 30\,k1\,Qr\,x^8\,y^2 + 60\,k1\,Qr\,x^6\,y^4 + 60\,k1\,Qr\,x^4\,y^6 + 30\,k1\,Qr\,x^2\,y^8 - 18\,\mu1\,Qr\,x^8\,y^2 - 10\,k1\,Fr\,x^4\,y^2 + 12\,x\,\mu1\,Qi\,y^9 + 6\,\mu1\,Qr\,y^{10} + Hr\,k1\,y^6 - 3\,k1\,Mr\,y^4 + 2\,k1\,Fr\,y^6 + Gr\,k1\,y^8 - 2\,\mu1\,Nr\,y^8 - 2\,\mu1\,Fr\,y^6 + 6\,k1\,Qr\,y^{10} - 3\,k1\,Mr\,x^4 - Hr\,x^6\,k1 + Gr\,x^8\,k1 + 2\,\mu1\,Fr\,x^6 - 6\,\mu1\,Qr\,x^{10} + 6\,k1\,Qr\,x^{10} - 2\,\mu1\,Nr\,x^8 + 2\,k1\,Fr\,x^6) \Big/ (2\,\mu1\,k1\,(x^2+y^2)^4)$$

```
> eyy1:=simplify(diff(uy1,y));
```

$$eyy1 := (-8\,\mu1\,Fi\,y^3\,x^3 - 4\,\mu1\,Fi\,y^5\,x + 12\,\mu1\,Qr\,x^6\,y^4 - 12\,\mu1\,Qr\,x^4\,y^6 - 18\,\mu1\,Qr\,x^2\,y^8 - Hr\,x^4\,k1\,y^2 + Hr\,x^2\,k1\,y^4 + 4\,Gr\,x^6\,k1\,y^2 + 6\,Gr\,x^4\,k1\,y^4 + 18\,k1\,Mr\,y^2\,x^2 - 12\,k1\,Mi\,y\,x^3 + 12\,k1\,Mi\,y^3\,x - 10\,k1\,Fr\,y^4\,x^2 + 8\,k1\,Fi\,y\,x^5 - 8\,k1\,Fi\,y^5\,x - 12\,\mu1\,Qi\,x^9\,y - 48\,\mu1\,Qi\,x^7\,y^3 - 72\,\mu1\,Qi\,x^5\,y^5 - 48\,\mu1\,Qi\,x^3\,y^7 + 4\,Gr\,x^2\,k1\,y^6 + 8\,\mu1\,Nr\,x^6\,y^2 + 12\,\mu1\,Nr\,x^4\,y^4 + 8\,\mu1\,Nr\,x^2\,y^6 - 2\,\mu1\,Fr\,x^4\,y^2 + 2\,\mu1\,Fr\,x^2\,y^4 - 4\,\mu1\,Fi\,y\,x^5 + 30\,k1\,Qr\,x^8\,y^2 + 60\,k1\,Qr\,x^6\,y^4 + 60\,k1\,Qr\,x^4\,y^6 + 30\,k1\,Qr\,x^2\,y^8 + 18\,\mu1\,Qr\,x^8\,y^2 - 10\,k1\,Fr\,x^4\,y^2 - 12\,x\,\mu1\,Qi\,y^9 - 6\,\mu1\,Qr\,y^{10} + Hr\,k1\,y^6 - 3\,k1\,Mr\,y^4 + 2\,k1\,Fr\,y^6 + Gr\,k1\,y^8 + 2\,\mu1\,Nr\,y^8 + 2\,\mu1\,Fr\,y^6 + 6\,k1\,Qr\,y^{10} - 3\,k1\,Mr\,x^4 - Hr\,x^6\,k1 + Gr\,x^8\,k1 - 2\,\mu1\,Fr\,x^6 + 6\,\mu1\,Qr\,x^{10} + 6\,k1\,Qr\,x^{10} + 2\,\mu1\,Nr\,x^8 + 2\,k1\,Fr\,x^6) \Big/ (2\,\mu1\,k1\,(x^2+y^2)^4)$$

```
> exy1:=simplify((diff(ux1,y)+diff(uy1,x))/2);
```

$$exy1 := (6\,Qi\,x^{10} + 2\,Fi\,y^6 + 6\,Qi\,y^{10} + 2\,Fi\,x^6 + Gi\,y^8 + Gi\,x^8 - 3\,Mi\,x^4 - 3\,Mi\,y^4 + 18\,Mi\,y^2\,x^2 + 30\,Qi\,y^8\,x^2 + 4\,Gi\,y^6\,x^2 + 4\,Gi\,y^2\,x^6 + 6\,Gi\,y^4\,x^4 + 2\,Hr\,x^5\,y + 4\,Hr\,x^3\,y^3 + 2\,Hr\,x\,y^5 + 12\,Mr\,x^3\,y - 12\,Mr\,x\,y^3 + 8\,Fr\,x\,y^5 - 10\,Fi\,x^4\,y^2 - 10\,Fi\,y^4\,x^2 + 60\,Qi\,x^4\,y^6 + 30\,Qi\,x^8\,y^2 + 60\,Qi\,x^6\,y^4 - 8\,y\,Fr\,x^5) \Big/ (2\,\mu1\,(x^2+y^2)^4)$$

```
> exx2:=simplify(diff(ux2,x));
```

$$exx2 := \frac{-6\,Wr\,x^2\,k2 - 6\,Wr\,y^2\,k2 + 2\,\mu2\,Ar + 6\,\mu2\,Wr\,x^2 - 6\,\mu2\,Wr\,y^2 - 12\,\mu2\,Wi\,x\,y - Br\,k2}{2\,\mu2\,k2}$$

```
> eyy2:=simplify(diff(uy2,y));
```

$$eyy2 := \frac{6\,Wr\,x^2\,k2 + 6\,Wr\,y^2\,k2 + 2\,\mu2\,Ar + 6\,\mu2\,Wr\,x^2 - 6\,\mu2\,Wr\,y^2 - 12\,\mu2\,Wi\,x\,y + Br\,k2}{2\,\mu2\,k2}$$

```
> exy2:=simplify((diff(ux2,y)+diff(uy2,x))/2);
```

$$exy2 := \frac{6\,Wi\,y^2 + \mathrm{Bi} + 6\,Wi\,x^2}{2\,\mu2}$$

```
> tre1:=factor(simplify(subs(x=rho*cos(theta),y=rho*sin(theta),sim
  plify(exx1+eyy1)))):
> tre2:=factor(simplify(subs(x=rho*cos(theta),y=rho*sin(theta),sim
  plify(exx2+eyy2)))):
> dete1:=factor(simplify(subs(x=rho*cos(theta),y=rho*sin(theta),si
  mplify(exx1*eyy1-exy1^2)))):
> dete2:=factor(simplify(subs(x=rho*cos(theta),y=rho*sin(theta),si
  mplify(exx2*eyy2-exy2^2)))):
> intetre2:=factor(simplify(int(int((tre2)^2*rho,theta=0..2*Pi),rh
  o=0..R))) assuming(R>0, Rt>R);
```

$$intetre2 := \frac{2\,\pi\,R^2\,(3\,R^4\,Wr^2 + 3\,R^4\,Wi^2 + 2\,Ar^2)}{k2^2}$$

```
> intetre1:=factor(simplify(int(int((tre1)^2*rho,theta=0..2*Pi),rh
  o=R..Rt))) assuming(R>0, Rt>R);
```

$$intetre1 := 2\,\pi\,(Rt - R)\,(R + Rt)\,(3\,Rt^6\,R^2\,Qr^2 + 3\,Rt^6\,R^2\,Qi^2 + 3\,Rt^4\,R^4\,Qr^2 + 3\,Rt^4\,R^4\,Qi^2 + 6\,Rt^2\,R^2\,Fi\,Qi - 6\,Rt^2\,R^2\,Qr\,Fr + 2\,Rt^2\,R^2\,Nr^2 + 3\,Rt^2\,R^6\,Qr^2 + 3\,Rt^2\,R^6\,Qi^2 + Fr^2 + Fi^2) \Big/ (R^2\,k1^2\,Rt^2)$$

```
> intedete2:=factor(simplify(int(int(dete2*rho,theta=0..2*Pi),rho=
  0..R)));
```

$$intedete2 := \pi\,R^2\,(6\,R^4\,\mu2^2\,Wr^2 - 12\,R^4\,Wr^2\,k2^2 + 6\,R^4\,\mu2^2\,Wi^2 - 12\,R^4\,k2^2\,Wi^2 - 6\,k2^2\,R^2\,Wr\,Br - 6\,k2^2\,R^2\,Wi\,\mathrm{Bi} + 4\,\mu2^2\,Ar^2 - Br^2\,k2^2 - k2^2\,\mathrm{Bi}^2) \Big/ (4\,\mu2^2\,k2^2)$$

```
> intedete1:=factor(simplify(int(int(dete1*rho,theta=0..2*Pi),rho=
  R..Rt))) assuming(R>0, Rt>R);
```

$$intedete1 := -\pi\,(Rt - R)\,(R + Rt)\,(6\,Rt^8\,R^6\,k1^2\,Gr\,Qr + 6\,Rt^8\,R^6\,k1^2\,Gi\,Qi + 12\,Rt^6\,R^6\,Fr\,Qr\,\mu1^2 - 12\,Rt^6\,R^6\,Fi\,Qi\,\mu1^2 + 6\,Rt^6\,R^8\,k1^2\,Gr\,Qr + 6\,Rt^6\,R^8\,k1^2\,Gi\,Qi - 6\,Rt^4\,k1^2\,Fi\,Mi\,R^2 - 6\,Rt^4\,k1^2\,Fr\,Mr\,R^2 - 6\,Rt^2\,R^4\,k1^2\,Fi\,Mi - 6\,Rt^2\,R^4\,k1^2\,Fr\,Mr + 12\,Rt^{10}\,R^6\,Qr^2\,k1^2 + 12\,Rt^{10}\,R^6\,Qi^2\,k1^2 - 6\,Rt^{10}\,R^6\,Qr^2\,\mu1^2 - 6\,Rt^{10}\,R^6\,Qi^2\,\mu1^2 + 12\,Rt^8\,R^8\,Qr^2\,k1^2 + 12\,Rt^8\,R^8\,Qi^2\,k1^2 - 6\,Rt^8\,R^8\,Qr^2\,\mu1^2 - 6\,Rt^8\,R^8\,Qi^2\,\mu1^2 + Rt^6\,R^6\,Gr^2\,k1^2 + Rt^6\,R^6\,Gi^2\,k1^2 - 4\,Rt^6\,R^6\,Nr^2\,\mu1^2 + 12\,Rt^6\,R^{10}\,Qr^2\,k1^2 + 12\,Rt^6\,R^{10}\,Qi^2\,k1^2 - 6\,Rt^6\,R^{10}\,Qr^2\,\mu1^2 - 6\,Rt^6\,R^{10}\,Qi^2\,\mu1^2 + Rt^4\,Hr^2\,k1^2\,R^4 + 4\,Rt^4\,Fi^2\,k1^2\,R^4 + 4\,Rt^4\,Fr^2\,k1^2\,R^4 - 2\,Rt^4\,Fi^2\,\mu1^2\,R^4 - 2\,Rt^4\,Fr^2\,\mu1^2\,R^4 + 3\,Rt^2\,R^2\,Mr^2\,k1^2 + 3\,Rt^2\,R^2\,Mi^2\,k1^2 + 3\,R^4\,Mr^2\,k1^2 + 3\,R^4\,Mi^2\,k1^2 + 3\,Rt^4\,Mi^2\,k1^2 + 3\,Rt^4\,Mr^2\,k1^2) \Big/ (4\,R^6\,\mu1^2\,k1^2\,Rt^6)$$

```
> intetre1cubo:=factor(simplify(int(int((tre1)^3*rho,theta=0..2*Pi
  ),rho=R..Rt))) assuming(R>0, Rt>R);
```

$$intetre1cubo := 4\,Nr\,\pi\,(Rt - R)\,(R + Rt)\,(9\,Rt^6\,R^2\,Qr^2 + 9\,Rt^6\,R^2\,Qi^2 + 9\,Rt^4\,R^4\,Qr^2 + 9\,Rt^4\,R^4\,Qi^2 - 18\,Rt^2\,R^2\,Qr\,Fr + 2\,Rt^2\,R^2\,Nr^2 + 18\,Rt^2\,R^2\,Fi\,Qi + 9\,Rt^2\,R^6\,Qr^2 + 9\,Rt^2\,R^6\,Qi^2 + 3\,Fi^2 + 3\,Fr^2) \Big/ (R^2\,k1^3\,Rt^2)$$

```
> intetre2cubo:=factor(simplify(int(int((tre2)^3*rho,theta=0..2*Pi
  ),rho=0..R))) assuming(R>0, Rt>R);
```

$$intetre2cubo := \frac{4\,Ar\,\pi\,R^2\,(9\,R^4\,Wi^2 + 9\,R^4\,Wr^2 + 2\,Ar^2)}{k2^3}$$

```
> intetrdet1:=factor(simplify(int(int((tre1*dete1)*rho,theta=0..2*
  Pi),rho=R..Rt))) assuming(R>0, Rt>R);
```

$$intetrdet1 := -\pi\,(Rt - R)\,(R + Rt)\,(-6\,Rt^2\,R^4\,k1^2\,Nr\,Fi\,Mi - 6\,Rt^2\,R^4\,k1^2\,Nr\,Fr\,Mr - Rt^2\,R^2\,Fi\,Hr\,Mi\,k1^2 - Rt^2\,R^2\,Fr\,Hr\,Mr\,k1^2 + 6\,Rt^8\,R^6\,k1^2\,Nr\,Gi\,Qi + 6\,Rt^8\,R^6\,k1^2\,Nr\,Gr\,Qr - 3\,Rt^6\,R^6\,Qi\,Gi\,Hr\,k1^2 + 36\,Rt^6\,R^6\,Nr\,Fr\,Qr\,\mu1^2 - 3\,Rt^6\,R^6\,Qr\,Gr\,Hr\,k1^2 - 36\,Rt^6\,R^6\,Nr\,Fi\,Qi\,\mu1^2 + 6\,Rt^6\,R^8\,k1^2\,Nr\,Gi\,Qi + 6\,Rt^6\,R^8\,k1^2\,Nr\,Gr\,Qr - Rt^4\,Fi\,Gi\,Hr\,k1^2\,R^4 + Rt^4\,Fr\,Gr\,Hr\,k1^2\,R^4 + 9\,Rt^4\,Qr\,Hr\,Mr\,k1^2\,R^4 - 9\,Rt^4\,Qi\,Hr\,Mi\,k1^2\,R^4 - 6\,Rt^4\,k1^2\,Nr\,Fi\,Mi\,R^2 - 6\,Rt^4\,k1^2\,Nr\,Fr\,Mr\,R^2 + Nr\,Hr^2\,k1^2\,R^4\,Rt^4 - R^4\,Fi\,Hr\,Mi\,k1^2 - R^4\,Fr\,Hr\,Mr\,k1^2 - 18\,Rt^{10}\,R^6\,Nr\,Qi^2\,\mu1^2 - 18\,Rt^{10}\,R^6\,Nr\,Qr^2\,\mu1^2 + 12\,Rt^{10}\,R^6\,Nr\,Qi^2\,k1^2 + 12\,Rt^{10}\,R^6\,Nr\,Qr^2\,k1^2 - 9\,Rt^8\,R^6\,k1^2\,Qr^2\,Hr - 9\,Rt^8\,R^6\,k1^2\,Qi^2\,Hr - 18\,Rt^8\,R^8\,Nr\,Qi^2\,\mu1^2 - 18\,Rt^8\,R^8\,Nr\,Qr^2\,\mu1^2 + 12\,Rt^8\,R^8\,Nr\,Qi^2\,k1^2 + 12\,Rt^8\,R^8\,Nr\,Qr^2\,k1^2 + Rt^6\,R^6\,Nr\,Gr^2\,k1^2 + Rt^6\,R^6\,Nr\,Gi^2\,k1^2 - 9\,Rt^6\,R^8\,k1^2\,Qr^2\,Hr - 9\,Rt^6\,R^8\,k1^2\,Qi^2\,Hr - 18\,Rt^6\,R^{10}\,Nr\,Qi^2\,\mu1^2 - 18\,Rt^6\,R^{10}\,Nr\,Qr^2\,\mu1^2 + 12\,Rt^6\,R^{10}\,Nr\,Qi^2\,k1^2 + 12\,Rt^6\,R^{10}\,Nr\,Qr^2\,k1^2 - 6\,Rt^4\,Nr\,Fi^2\,\mu1^2\,R^4 + 4\,Rt^4\,Nr\,Fr^2\,k1^2\,R^4 - 6\,Rt^4\,Nr\,Fr^2\,\mu1^2\,R^4 + 4\,Rt^4\,Nr\,Fi^2\,k1^2\,R^4 + Rt^4\,k1^2\,Fr^2\,Hr\,R^2 + Rt^4\,k1^2\,Fi^2\,Hr\,R^2 - Rt^4\,Fi\,Hr\,Mi\,k1^2 - Rt^4\,Fr\,Hr\,Mr\,k1^2 + 3\,Rt^2\,R^2\,Nr\,Mr^2\,k1^2 + 3\,Rt^2\,R^2\,Nr\,Mi^2\,k1^2 + Rt^2\,R^4\,k1^2\,Fr^2\,Hr + Rt^2\,R^4\,k1^2\,Fi^2\,Hr + 3\,Rt^4\,Nr\,Mi^2\,k1^2 + 3\,Rt^4\,Nr\,Mr^2\,k1^2 + 3\,R^4\,Nr\,Mi^2\,k1^2 + 3\,R^4\,Nr\,Mr^2\,k1^2 - 4\,R^6\,Nr^3\,\mu1^2\,Rt^6)\,/\,(2\,R^6\,k1^3\,\mu1^2\,Rt^6)$$

```
> intetrdet2:=factor(simplify(int(int((tre2*dete2)*rho,theta=0..2*
  Pi),rho=0..R))) assuming(R>0, Rt>R);
```

$$intetrdet2 := Ar\,\pi\,R^2\,(-12\,R^4\,Wr^2\,k2^2 - 12\,R^4\,k2^2\,Wi^2 + 18\,R^4\,\mu2^2\,Wi^2 + 18\,R^4\,\mu2^2\,Wr^2 - 6\,k2^2\,R^2\,Wr\,Br - 6\,k2^2\,R^2\,Wi\,\mathrm{Bi} + 4\,\mu2^2\,Ar^2 - Br^2\,k2^2 - k2^2\,\mathrm{Bi}^2)\,/\,(2\,k2^3\,\mu2^2)$$

```
> ene:=simplify((1/2)*(k1+mu1)*intetre1-2*mu1*intedete1+(1/2)*(k2+
  mu2)*intetre2-2*mu2*intedete2):
> ene2:=simplify(3*(e1+f1)*intetre1cubo-6*e1*intetrdet1+3*(e2+f2)*
  intetre2cubo-6*e2*intetrdet2):
> c:=R^2/Rt^2;
```

$$c := \frac{R^2}{Rt^2}$$

```
> Ar:=(epsilon11+epsilon22)*(chi1+1)/((2*c+chi1-1)*(chi2-1)/mu2+2*
  (1-c)*(chi1-1)/mu1):
> Nr:=(epsilon11+epsilon22)*(mu1/mu2*(chi2-1)+2)/((2*c+chi1-1)*(ch
  i2-1)/mu2+2*(1-c)*(chi1-1)/mu1):
> Hr:=2*c*(epsilon11+epsilon22)*(chi1-1-mu1/mu2*(chi2-1))/((2*c+ch
  i1-1)*(chi2-1)/mu2+2*(1-c)*(chi1-1)/mu1)*Rt^2:
> alpha:=(chi1/mu1-chi2/mu2)-chi1*(1/mu1+chi2/mu2)/c^3:
> beta:=(chi1/mu1+1/mu2)-chi1*c*(1/mu1-1/mu2):
```

```
> provacmu:=(1/mu1-1/mu2)*mu1*(epsilon11-epsilon22+2*I*epsilon12)*
  R^2/(-3*(1/mu1+chi2/mu2)*(1/mu1-1/mu2)*(1-1/c)^2/alpha-beta):
> Fr:=Re(provacmu) assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> Fi:=Im(provacmu)assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> provac3:=-conjugate(provacmu)*(1/mu1+chi2/mu2)*(1-1/c)/alpha/R^4
  assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> Qr:=Re(provac3)assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> Qi:=Im(provac3)assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> provadm3:=chi1*conjugate(provac3)*Rt^6+(provacmu)*Rt^2 assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> Mr:=Re(provadm3)assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> Mi:=Im(provadm3)assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> provad1:=chi1*conjugate(provacmu)/Rt^2-3*(provac3)*Rt^2-mu1*(eps
  ilon11-epsilon22-2*I*epsilon12) assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> Gr:=Re(provad1) assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> Gi:=Im(provad1) assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> provaa3:=provac3*(1+chi1/c^3)-conjugate(provacmu)*(1-1/c)/R^4
  assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> Wr:=Re(provaa3)assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> Wi:=Im(provaa3)assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
```

```
   ,epsilon12,real,R,real,Rt,real):
> provab1:=conjugate(provacmu)*((4-3/c)/R^2+chi1/Rt^2)-3*provac3*(
  Rt^2+chi1*R^2/c^3)-mu1*(epsilon11-epsilon22-2*I*epsilon12)
  assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> Br:=Re(provab1)assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> Bi:=Im(provab1)assuming
  (mu1,real,mu2,real,k1,real,k2,real,epsilon11,real,epsilon22,real
  ,epsilon12,real,R,real,Rt,real):
> kk1:=10;kk2:=30;mumu1:=15;mumu2:=20;ee1:=1;ee2:=3;ff1:=4;ff2:=2;
```

$$kk1 := 10$$

$$kk2 := 30$$

$$mumu1 := 15$$

$$mumu2 := 20$$

$$ee1 := 1$$

$$ee2 := 3$$

$$ff1 := 4$$

$$ff2 := 2$$

```
> ene12:=factor(simplify(subs(epsilon11=0,epsilon22=0,epsilon12=1,
  k1=kk1,mu1=mumu1,k2=kk2,mu2=mumu2,R=p^(1/2),Rt=1,ene))):
> ene11:=factor(simplify(subs(epsilon11=1,epsilon22=0,epsilon12=0,
  k1=kk1,mu1=mumu1,k2=kk2,mu2=mumu2,R=p^(1/2),Rt=1,ene))):
> mueff:=factor(simplify(ene12/(2*Pi)));keff:=factor(simplify(2*en
  e11/(Pi)-mueff));
```

$$mueff := -\frac{15\,(9\,p^4 + 204\,p^3 - 66\,p^2 - 11\,p - 836)}{-204\,p^3 + 66\,p^2 - 209\,p + 836 + 36\,p^4}$$

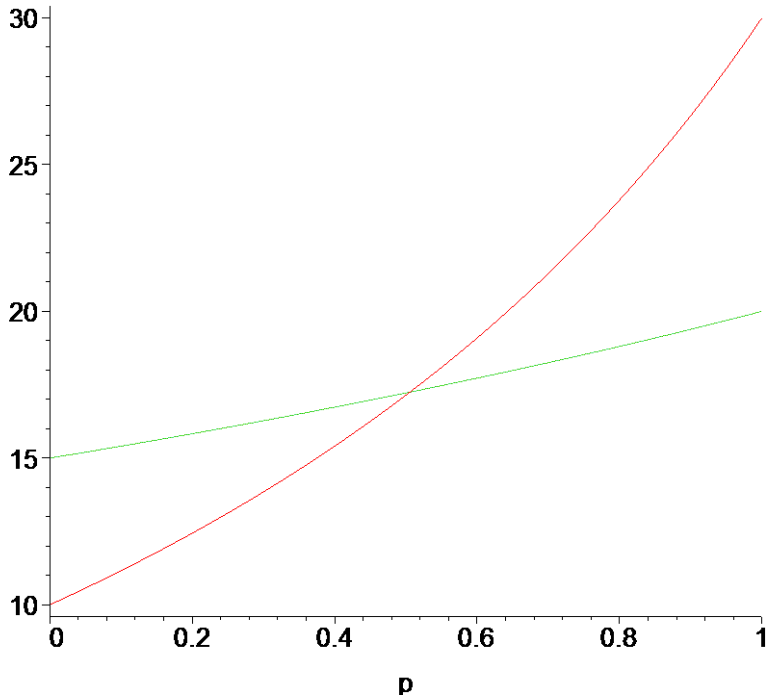
$$keff := -\frac{30\,(2\,p + 3)}{4\,p - 9}$$

```
> enex:=factor(simplify(subs(epsilon11=1,epsilon22=0,epsilon12=0,k
  1=kk1,mu1=mumu1,k2=kk2,mu2=mumu2,e1=ee1,e2=ee2,f1=ff1,f2=ff2,R=p
  ^(1/2),Rt=1,ene2/Pi/3))):
> eney:=factor(simplify(subs(epsilon11=1,epsilon22=1,epsilon12=0,k
  1=kk1,mu1=mumu1,k2=kk2,mu2=mumu2,e1=ee1,e2=ee2,f1=ff1,f2=ff2,R=p
  ^(1/2),Rt=1,ene2/Pi/12))):
> feff:=factor(simplify(eney-enex));eeff:=factor(simplify(2*enex-e
  ney));
```

$$feff := -\,(-1278911715\,p^5 + 495543414\,p^6 - 11802609104\,p - 4426937438\,p^3 + 3883438185\,p^4 + 5456854249\,p^2 + 8151922944 + 2957400\,p^9 + 3189456\,p^{10} - 184829364\,p^7 - 24993027\,p^8) \Big/ (4\,(-204\,p^3 + 66\,p^2 - 209\,p + 836 + 36\,p^4)^2\,(4\,p - 9)^3)$$

$$eeff := 3\,(74223\,p^8 + 606756\,p^7 - 1633830\,p^6 + 196071\,p^5 - 1992485\,p^4 + 7256282\,p^3 - 1521091\,p^2 - 1548800\,p - 4193376) \Big/ (2\,(4\,p-9)\,(-204\,p^3 + 66\,p^2 - 209\,p + 836 + 36\,p^4)^2)$$

```
> plot({mueff,keff},p=0..1);
```




```
> plot({eeff,feff},p=0..1);
```