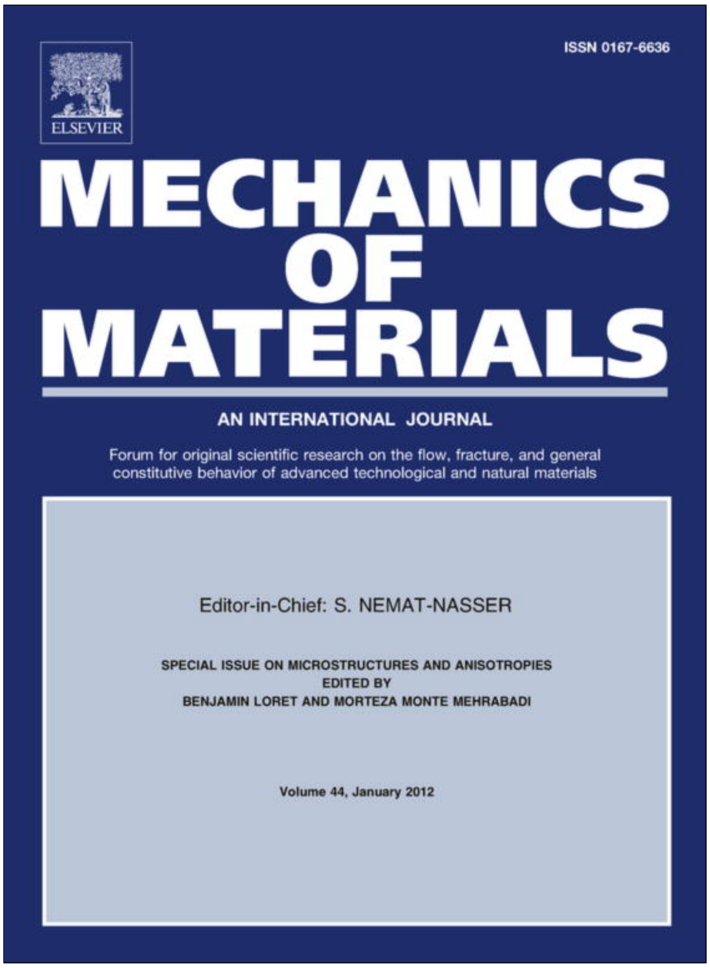
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Elastic behavior of inhomogeneities with size and shape different from their hosting cavities

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ABSTRACT

In this paper we consider an application of the Eshelby theory concerning the elastic behavior of prestrained or prestressed inhomogeneities. The theory, in its original version, deals with a configuration where both the ellipsoidal particle and the surrounding matrix are in elastostatic equilibrium if no external loads are applied to the system. Here, we consider slightly different shapes and sizes for the particle and the hosting cavity (whose surfaces are firmly bonded together) and, therefore, we observe a given state of strain (or stress) even without externally applied loads. We develop a complete procedure able to determine the uniform elastic field induced in an arbitrarily prestrained particle subjected to arbitrary remote loadings.

1. Introduction

Probably the most important result in the extensive literature on elastic composites is the Eshelby theorem on the response of a single isotropic ellipsoidal elastic particle in an isotropic elastic space subjected to a remote strain. Eshelby (1957, 1959, 1961) proved that an applied uniform strain results in a uniform strain within the ellipsoidal inhomogeneity. The so-called Eshelby tensor, involved in the strain calculation, depends on the geometry of the ellipsoid (i.e. on the ratios of semi-axes lengths) and on the elastic properties of the homogeneous hosting matrix. Some years later, a similar property was proved also for the elastic field of an anisotropic particle embedded in an anisotropic medium (Walpole, 1967).

The Eshelby solutions, in their first version, have been found within the elastostatics regime. The dynamic Eshelby inclusion problem for an ellipsoidal inclusion in a three-dimensional isotropic medium was recently considered (Michelitsch et al., 2003). The dynamic Eshelby tensor has been expressed in terms of solutions of the Helmholtz equation. This approach leads to closed-form expressions in the particular cases of spheres and cylinders coinciding with those given by Mikata and Nemat-Nasser (1990) and by Cheng and Batra (1999), obtained by employing different techniques.

Two different problems concerning the elastic fields generated by particular inhomogeneities have been solved by Walpole (1991a,b). In the first case a rigid inhomogeneity of ellipsoidal shape is bonded firmly at arbitrary orientation to a surrounding, unbounded, homogeneous matrix, and is translated infinitesimally by the action of an externally imposed force (Walpole, 1991a). A second problem deals with a rigid inclusion of ellipsoidal shape, bonded firmly at arbitrary orientation to a surrounding matrix, and rotated infinitesimally, about an axis through its center, by means of an externally imposed couple (Walpole, 1991a).

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Some other interesting generalizations have been made to obtain the counterpart of the (elastic) Eshelby property for different physical situations. Berryman (1997) has shown how the Eshelby theory may be generalized to both poroelasticity and thermoelasticity. The resulting formulas are important for applications to analysis of poroelastic and thermoelastic composites (Dormieux et al., 2006). Moreover, a simple and unified explicit expression for piezoelectric Eshelby tensors is presented by Huang and Yu (1994). Furthermore, the magneto-electro-elastic Eshelby tensor that describes the stress, the electric displacement and the magnetic induction in a piezomagnetic-piezoelectric composite has been introduced (Huang et al., 1998). The electric version of the Eshelby theory for an arbitrarily anisotropic environment has been recently developed by Giordano and Palla (2008).

The most important aspect of the Eshelby work is that the interior points Eshelby tensor is constant for an ellipsoidal inhomogeneity. This fact implies that an uniform strain at infinity results in a uniform strain in the ellipsoidal inhomogeneity. Eshelby (1961) conjectured that, among all the closed surfaces, the ellipsoid alone has this convenient property. The strong Eshelby conjecture is: if the induced elastic fields inside an inhomogeneity are uniform under a single uniform loading, the inhomogeneity is of elliptic or ellipsoidal shape. The weak Eshelby conjecture is: if the induced elastic fields inside an inhomogeneity are uniform under all (any) uniform loadings, the inhomogeneity is of elliptic or ellipsoidal shape. Of course, the strong conjecture implies the weak conjecture. The first result was found by Sendeckyj (1970) who proved the strong Eshelby conjecture for two-dimensional inhomogeneities (plane strain or plane stress conditions). Successively, Ru and Schiavone (1996) verified the strong Eshelby conjecture for anti-plane problem. More recently, Kang and Milton (2008) proved the weak Eshelby conjecture for three-dimensional inclusions for isotropic materials by using the maximum principle of harmonic potentials. Finally, Liu (2008) showed that the strong Eshelby conjecture is false in three and higher dimensions, by constructing explicit counterexamples.

The Eshelby result and its generalizations have been found to be useful in the analysis of composite materials: the crucial point is the determination of the effective physical properties exhibited at the macroscopic scale (Van Beek, 1967; Walpole, 1981). The homogenization procedures contain at first the exact mathematical analysis of the mechanical behavior induced by a single inhomogeneity (Mura, 1987; Nemat-Nasser and Hori, 1999), and then proceed by considering the more general case of interacting particles (Hashin, 1983; Markov, 2000). This approach is generally carried out in the limit of a low density defect population (Mori and Tanaka, 1973). Such an hypothesis can be partially removed by means of different methods, such as iterated homogenizations (Avellaneda, 1987) and differential schemes (McLaughlin, 1977; Giordano, 2003,). These techniques have been applied with great accuracy both to the case of embedded inhomogeneities (Hill, 1963; Snyder and Garboczi, 1992; Kachanov and Sevostianov, 2005) and to the case of dispersed defects, such as micro-cracks in a matrix (Budiansky and O'Connell, 1976; Kachanov, 1992; Giordano and Colombo, 2007a,b). The application of the Eshelby theory to evaluate the distribution of the elastic and electric fields around a crack has conducted to the definition of the densities of states for such quantities (Giordano, 2007; Giordano and Colombo, 2007c).

The mechanical behavior of nanostructured materials is strongly affected by interface features, occurring at the boundary between phases characterized by different elastic constitutive equations or crystalline structures (Palla et al., 2008, 2009, 2010). In particular, the embedding of a given nano-inhomogeneity in a hosting matrix is deeply influenced by the lattice mismatch and by the possible differences between the external surface of the particle and the internal surface of the hosting cavity. In fact, both the inhomogeneity and the matrix accomplish an elastic relaxation to accommodate these mismatches and, therefore, they admit a state of deformation even if no external load is applied. We will refer to such a complex system as a prestrained (or, equivalently, prestressed) composite. In particular, in this work, we develop a mathematical procedure able to quantify the prestrains (or prestresses) induced by the differences between the particle and the cavity in a continuum. It means that we analyse the deformations necessary to create the perfect bonding between the external surface of the particle and the internal surface of the cavity.

In general, the results of the present paper can be applied every time we deal with an inhomogeneity having extent and geometry somewhat different from the hole in the matrix. We are also able to take into account an arbitrary uniform mechanical load remotely applied to the system. Throughout the paper we introduce and solve the above problem within the geometrical framework of the small strain elasticity theory. Therefore, the results can be applied to any physical situation (independently of the specific materials or the macro-, micro-or nano-scale) which can be represented by the above introduced general scheme.

Nevertheless, an interesting example of application is given by composite materials and structures at the nano-scale. In Fig. 1 one can find the atomistic structure of an interface between a matrix and a cylindrical

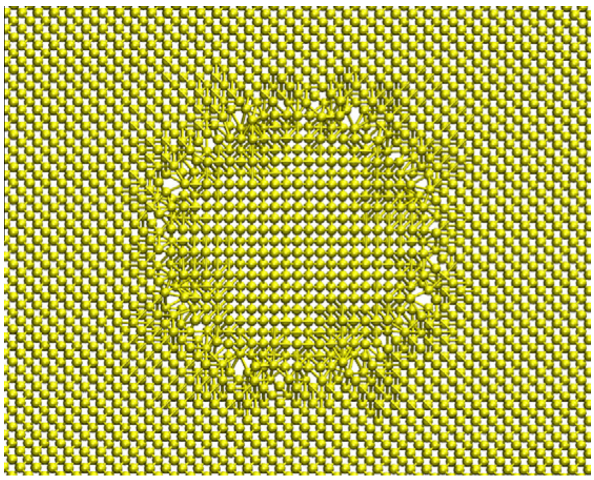

Fig. 1. Example of an atomistically resolved prestrained cylindrical inhomogeneity.

nano-inhomogeneity: the effects of the lattice mismatch between cylinder and cavity are evident in the region close to the interface. That zone is in fact characterized by a disorder (with atoms density larger than the pure crystal phase) which generates a quite uniform hydrostatic compression within the particle. This effect can be simply modelled through the insertion of a particle with radius larger than the matrix cavity (Palla et al., 2009). A very similar situation is found when the particle exhibits a thermal coefficient different from the matrix one (Wakashima et al., 1974). In all these cases the particle experiences an eigenstrain which leads to nonzero stress also without external loads. While previous works (Dvorak, 1992) discussed incremental thermomechanical loads and transformation strains in the phases, here we analyse the eigenstrain induced by the different geometry between particle and matrix. In other words, other than the typical eigenstrain introduced by Eshelby to take into account the elastic contrast between particle and matrix, we determine the additional eigenstrain induced by the geometrical (size and shape) contrast between the inhomogeneity and the cavity.

Typical examples of prestressed systems in recent nanotechnology are represented by semiconductor quantum dots or quantum wires, embedded in a matrix with different properties. Several works have been addressed to the calculation of the strain state in buried quantum dots (Sharma and Ganti, 2002; Zhang and Sharma, 2005). Both quantitative and qualitative knowledge of stress and strain distributions are essential for characterizing and tailoring their optoelectronic properties (Singh, 1992; Maranganti and Sharma, 2006), as well as for understanding their self-organization (Timm et al., 2008). Typically, the state of deformation is estimated using continuum elasticity and, then, used as input for an electronic structure calculation (Harrison, 2005). However, while continuum elasticity is inherently scale-independent, the elastic relaxation of a nanostructure does depend on the actual length scale at which the heterogeneity shows up (Duan et al., 2005,, 2008). In other words, at the nanoscale surface effects become important due to the increasing surface-to-volume ratio and induce a size dependency in the overall elastic behavior (Sharma et al., 2003; Sharma and Ganti, 2004).

In our approach, first of all we will analyse inhomogeneities with circular symmetry, namely cylinders and spheres of radius R_2, embedded in a matrix with a cavity of different radius R_1. The radius difference is considered very small, thus allowing for the application of the infinitesimal theory of elasticity (Landau and Lifschitz, 1959; Atkin and Fox, 2005). It is important to remark that this configuration corresponds to a continuum dislocation distributed over the (cylindrical or spherical) interface between the materials (Willis, 1965; Eshelby, 1973). It must be considered as a Volterra dislocation with constant Burger vector of modulus $R_1 - R_2$ and radial direction (referred to as $\vec{n}$). More specifically, if we start from a situation with a not prestrained inhomogeneity and we consider a dislocation at the interface with Burger vector $\vec{b} = (R_1 - R_2)\vec{n}$, we obtain the final configuration corresponding to the prestrained inhomogeneity. In this work we approach this problem with the theory of the inhomogeneities, based on the Eshelby tensor. The solution through the dislocation theory is much more complicated since we are dealing with a heterogeneous structure. The same problem is solved for an ellipsoidal inhomogeneity embedded in a different ellipsoidal cavity of the matrix. Also in this case there is a direct correspondence with the dislocation theory. In particular, the elastic fields can be attributed to a Somigliana dislocation distributed over the interface. In such a case the Burger vector connects a point of the inhomogeneity surface (in elastostatic equilibrium) with the corresponding point of the cavity surface in the matrix.

We want to remark that, in this work, we have analysed the fundamental micromechanical problem of a single prestrained particle and we have postponed the study of the effective behavior of a dispersion of prestrained inhomogeneities to future investigations. Nevertheless, the knowledge of the exact elastic fields inside a single particle allows to apply standard homogenization theories (dilute dispersions), as above discussed (Markov, 2000). It can be useful to observe that the presence of the geometrical eigenstrain is analogous to the presence of a pressure within saturated pores and, therefore, the homogenization methods used in linear microporoelasticity can be adopted as well (Dormieux et al., 2006). Since we may expect that the geometrical eigenstrain is dissimilar for different particles of a real dispersion, the above techniques should be applied by considering a different behavior for all the inhomogeneities. For high values of the volume fraction of particles the consideration of the exact interactions among them is very complicated and some approximated schemes (iterative, differential, self-consistent) could be applied.

The structure of the paper is the following: in Section 2 we give a brief outline of the Eshelby theory for an ellipsoidal inclusion. In Section 3 we introduce the generalized equivalence principle for spheres and cylinders and in Section 4 for ellipsoids. Conclusions and appendices with some mathematical details close the paper.

2. Outline of the Eshelby theory for ellipsoidal inclusions

The main purpose of this Section is to define the basic equations describing the elastic field inside and outside an ellipsoidal inclusion Ω embedded into a homogeneous matrix. The materials are supposed to be linear elastic, homogeneous and isotropic. We consider an infinite medium with stiffness tensor $\hat{\mathcal{C}}^{(1)}$; it means that the homogeneous solid matrix (hereafter labelled as material 1) is characterized by the relation $\hat{T} = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}$ or, in components, $T_{ij} = \mathcal{C}^{(1)}_{ijkh}\epsilon_{kh}$ where $\hat{T}$ is the stress tensor (with components T_{ij}), $\hat{\epsilon}$ is the strain tensor (with components ϵ_{ij}). Moreover, we define an embedded ellipsoidal inclusion Ω as a region of space described by the constitutive equation $\hat{T} = \hat{\mathcal{C}}^{(1)}(\hat{\epsilon} - \hat{\epsilon}^*)$. The strain $\hat{\epsilon}^*$ is *a priori* given and it is called eigenstrain (or stress-free strain). In other words, throughout this paper we denote as an inclusion a region containing a distribution of eigenstrain with the same moduli as the matrix. It is important to remark that the concept of inclusion is different from that of inhomogeneity. The inhomogeneity is defined as follows: we consider an infinite medium with stiffness tensor $\hat{\mathcal{C}}^{(1)}$in $\Re^3 \setminus \Omega$ (matrix) and $\hat{\mathcal{C}}^{(2)}$ in the ellipsoidal region Ω (inhomogeneity). We remotely load the system with a uniform strain $\hat{\epsilon}^{\infty}$or, equivalently, with the uniform stress $\hat{T}^{\infty} = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{\infty}$. This configuration can be analysed by means of the Eshelby equivalence principle as discussed in the following Sections. For an isotropic matrix the stiffness tensor can be represented as

$$\mathcal{C}^{(1)}_{ijkh} = \left(K_1 - \frac{2}{3}\mu_1\right)\delta_{ij}\delta_{kh} + \mu_1(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}) \tag{1}$$

where the elastic moduli are named K_1(bulk modulus) and μ_1(shear modulus) and δ_{ij} is the kronecker delta. The bulk and the shear moduli can be replaced when needed by the Young modulus $E_1 = (9K_1\mu_1)/(\mu_1 + 3K_1)$ and the Poisson ratio $\nu_1 = (3K_1 - 2\mu_1)/(2(\mu_1 + 3K_1))$. The displacement u_i induced by the presence of the inclusion (i.e. of the uniform eigenstrain $\hat{\epsilon}^*$) can be evaluated in term of the so-called harmonic potential $\Phi(\vec{r})$ and biharmonic potential $\Psi(\vec{r})$ (Eshelby, 1957, 1959):

$$u_i(\vec{r}) = \epsilon^*_{kh}\left[\frac{1}{8\pi(1-\nu_1)}\Psi_{,ikh} - \frac{\delta_{ih}}{4\pi}\Phi_{,k} - \frac{\delta_{ik}}{4\pi}\Phi_{,h} - \frac{\nu_1}{1-\nu_1}\frac{\delta_{kh}}{4\pi}\Phi_{,i}\right] \tag{2}$$

where $\vec{r} = (x_1, x_2, x_3)$ is the position vector. Hereafter we write the symbol $f_{,i} = \frac{\partial f}{\partial x_i}$ and we extend this notation to higher order derivatives. Eq. (2) is valid anywhere. The harmonic potential is defined, as well known, by the Poisson equation $\nabla^2\Phi = -4\pi$ if $\vec{r} \in \Omega, 0$ if $\vec{r} \notin \Omega$ and the integral form of its solution is $\Phi(\vec{r}) = \int_\Omega \frac{1}{\|\vec{r}-\vec{x}\|} d\vec{x}$ where the symbol $\|\cdot\|$ indicates the standard Euclidean norm. Similarly, the biharmonic potential is defined by means of the biharmonic equation $\nabla^4\Psi = -8\pi$ if $\vec{r} \in \Omega, 0$ if $\vec{r} \notin \Omega$ and the standard integral representation is $\Psi(\vec{r}) = \int_\Omega \|\vec{r} - \vec{x}\| d\vec{x}$ (Eshelby, 1957; Mura, 1987).

Such harmonic and biharmonic potentials only contain geometrical information about the embedded ellipsoid (i.e. the semi-axes lengths b_1, b_2 and b_3). It is worthwhile recalling some explicit expressions providing the above potentials or their derivative as used to determine the elastic fields (Mura, 1987):

$$\begin{cases} \Phi(\vec{r}) = \pi b_1 b_2 b_3 \int_{\eta(\vec{r})}^{+\infty} \frac{1-f(\vec{r},s)}{R(s)} ds \\ \Psi_{,i}(\vec{r}) = \pi b_1 b_2 b_3 x_i \int_{\eta(\vec{r})}^{+\infty} \frac{1-f(\vec{r},s)}{R(s)} \frac{s}{b_i^2+s} ds \end{cases} \tag{3}$$

where $f(\vec{r}, s), \eta(\vec{r})$ and $R(s)$ are defined as follows:

$$\begin{cases} f(\vec{r}, s) = \frac{x_1^2}{b_1^2+s} + \frac{x_2^2}{b_2^2+s} + \frac{x_3^2}{b_3^2+s} \\ \eta(\vec{r}): \quad f(\vec{r}, \eta(\vec{r})) = 1 \\ R(s) = \sqrt{\left(b_1^2+s\right)\left(b_2^2+s\right)\left(b_3^2+s\right)} \end{cases} \tag{4}$$

The quantity $\eta(\vec{r})$ is defined in implicit form and it is considered as the largest positive root of the equation $f(\vec{r}, \eta(\vec{r})) = 1$. The integrals defined in Eq. (3) are used for the external region assuming $\eta(\vec{r})$ given in Eq. (4) and for the internal region assuming $\eta(\vec{r}) = 0$. We summarize the solution of the problem in terms of the gradient of the displacement and of the strain tensor. The gradient of the displacement is given by $J_{ij} = u_{i,j}$ and the strain tensor is defined as $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$. They can be evaluated accordingly to the relations

$$J_{ij} = \frac{\partial u_i}{\partial x_j} = \mathcal{D}_{ijkh}(\vec{r})\epsilon^*_{kh} \tag{5}$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \mathcal{S}_{ijkh}(\vec{r})\epsilon^*_{kh} \tag{6}$$

where $\mathcal{S}_{ijkh}(\vec{r})$ is the Eshelby tensor and $\mathcal{D}_{ijkh}(\vec{r})$ is a new tensor useful to determine the gradient of the displacement over the whole space. We observe that $\mathcal{S}_{ijkh}$ is the symmetrization of the tensor $\mathcal{D}_{ijkh}$ with respect to the first couple of indexes: $\mathcal{S}_{ijkh} = \frac{1}{2}(\mathcal{D}_{ijkh} + \mathcal{D}_{jikh})$. The generic forms of such tensors, which apply both inside and outside the inclusion, can be written by means of the elastic potentials as follows

$$\mathcal{S}_{ijkh}(\vec{r}) = \frac{1}{8\pi(1-\nu_1)}\Psi_{,ijkh} - \frac{\nu_1}{1-\nu_1}\frac{\delta_{kh}}{4\pi}\Phi_{,ij} - \frac{1}{8\pi}(\delta_{ih}\Phi_{,jk} + \delta_{ik}\Phi_{,jh} + \delta_{jh}\Phi_{,ik} + \delta_{jk}\Phi_{,ih}) \tag{7}$$

$$\mathcal{D}_{ijkh}(\vec{r}) = \frac{1}{8\pi(1-\nu_1)}\Psi_{,ijkh} - \frac{\nu_1}{1-\nu_1}\frac{\delta_{kh}}{4\pi}\Phi_{,ij} - \frac{1}{4\pi}(\delta_{ih}\Phi_{,kj} + \delta_{ik}\Phi_{,hj}) \tag{8}$$

Typically, the notation adopted for the tensors is different for the *internal* points and for *external* ones:

$$\begin{matrix} \mathcal{S}_{ijkh}(\vec{r}) = \mathcal{S}_{ijkh} & \mathcal{D}_{ijkh}(\vec{r}) = \mathcal{D}_{ijkh} & \text{if } \vec{r} \in \Omega \\ \mathcal{S}_{ijkh}(\vec{r}) = \mathcal{S}^{\infty}_{ijkh}(\vec{r}) & \mathcal{D}_{ijkh}(\vec{r}) = \mathcal{D}^{\infty}_{ijkh}(\vec{r}) & \text{if } \vec{r} \notin \Omega \end{matrix} \tag{9}$$

Taking a different notation for the *internal* and the *external* region is particularly efficient in order to remind that the internal tensors are constant and, therefore, the internal stress, strain and gradient of displacement are uniform tensor fields. By defining the depolarization factors of the first kind as

$$\Gamma_i = \frac{b_1 b_2 b_3}{2}\int_0^{+\infty} \frac{ds}{\left(b_i^2+s\right)R(s)} ds \tag{10}$$

and the depolarization factors of the second kind

$$\Theta_{ij} = \frac{b_1 b_2 b_3}{2}\int_0^{+\infty} \frac{sds}{(b_i^2+s)(b_j^2+s)R(s)} ds \tag{11}$$

we obtain the explicit expressions for the derivatives of the elastic potentials within the region Ω

$$\Phi_{,ij} = -4\pi\delta_{ij}\Gamma_i \tag{12}$$

$$\Psi_{,ijkh} = -4\pi(\delta_{ij}\delta_{kh}\Theta_{ki} + \delta_{ik}\delta_{jh}\Theta_{hi} + \delta_{ih}\delta_{jk}\Theta_{ji}) \tag{13}$$

Therefore, the internal tensors assume the explicit forms

$$\mathcal{D}_{ijkh} = -\frac{1}{2(1-\nu_1)}(\delta_{ij}\delta_{kh}\Theta_{ki} + \delta_{ik}\delta_{jh}\Theta_{hi} + \delta_{ih}\delta_{jk}\Theta_{ji}) + \frac{\nu_1}{1-\nu_1}\delta_{kh}\delta_{ij}\Gamma_i + \delta_{ih}\delta_{kj}\Gamma_k + \delta_{ik}\delta_{hj}\Gamma_h \tag{14}$$

$$\mathcal{S}_{ijkh} = \frac{1}{2}(\mathcal{D}_{ijkh} + \mathcal{D}_{jikh}) \tag{15}$$

The tensor $\mathcal{D}_{ijkh}$ is one of the most important quantities for the following derivations.

3. Generalized equivalence principle for spheres and cylinders

Eshelby has considered the case of a non-prestrained inhomogeneity embedded in a matrix which is remotely loaded by a given stress. This problem has been solved by means of the Eshelby equivalence principle according to which the complete problem is solved by combining the solutions of two different simpler problems (see Fig. 2). The first one (subproblem A) corresponds to the application of the remote uniform load to a simple homogeneous elastic system with the stiffness tensor of the matrix. The second configuration (subproblem B) corresponds to an unloaded inclusion (see previous Section) with a suitable eigenstrain. We follow the same technique but we introduce the generalized equivalence principle in order to consider a possible prestrain or prestress. Basically, in the generalized equivalence principle one must consider the constitutive equation of the inhomogeneity written in the reference frame corresponding to the configuration in which the particle fits exactly the cavity.

The first step in considering inhomogeneities with shape and size slightly different from the hosting cavity is given by the analysis of spherical or cylindrical particles embedded in cavities with different radii. A following Section will deal with the most general case of an ellipsoidal particle embedded in a different ellipsoidal cavity. More precisely, in this Section, we consider a (spherical or cylindrical) particle of radius R_2 and stiffness $\hat{C}^{(2)}$ which must be enclosed in the (spherical or cylindrical) cavity of radius R_1 in a matrix with

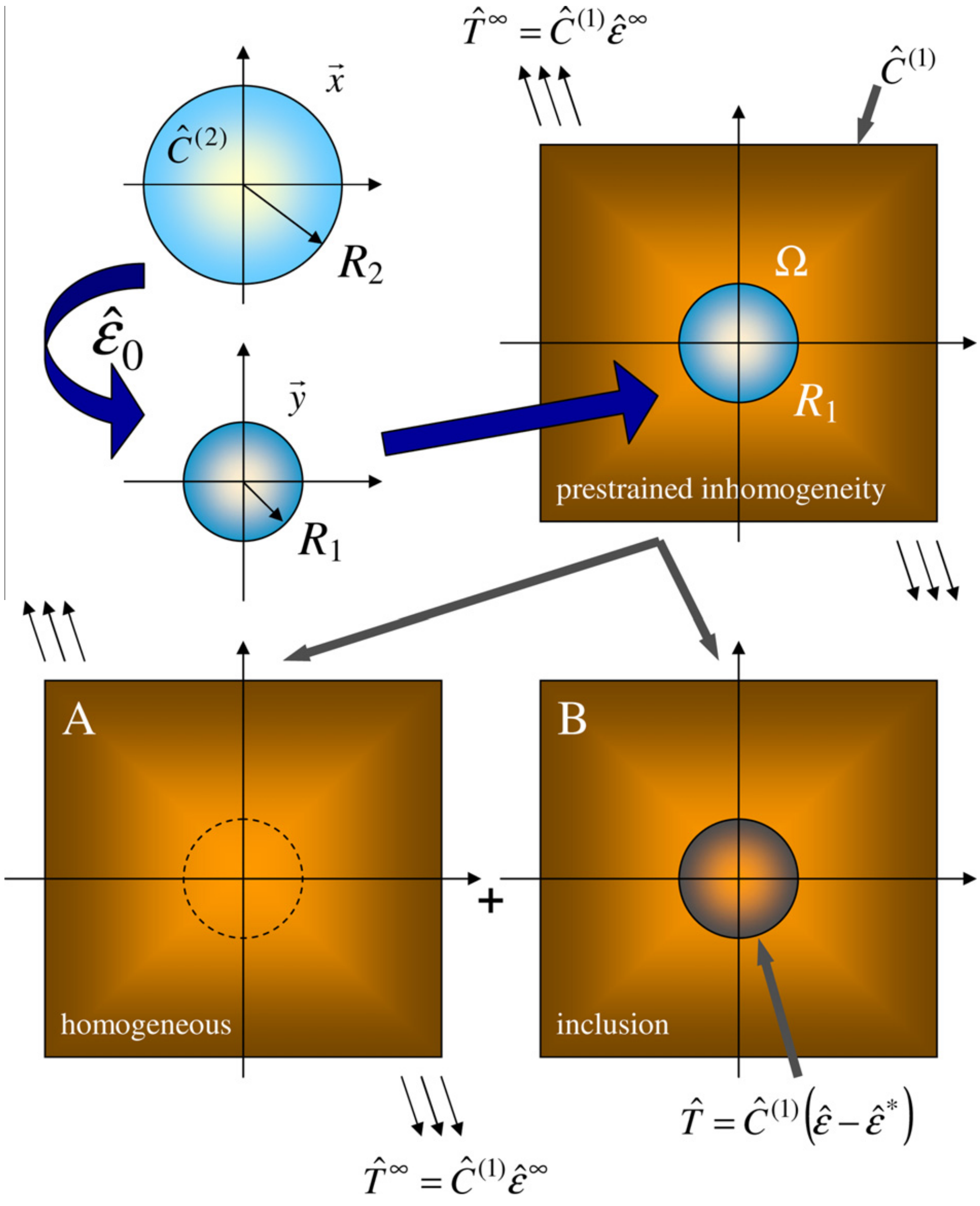

Fig. 2. Scheme of a prestrained (cylindrical or spherical) inhomogeneity (stiffness $\hat{C}^{(2)}$ and radius R_2) embedded into a homogeneous matrix (stiffness $\hat{C}^{(1)}$ and cavity with radius R_1). One can see the initial deformation $\hat{\epsilon}_0$ (applied to fit the cavity) and the representation of the generalized equivalence principle: the embedded particle can be studied through the superimposition of the subproblems *A* and *B* corresponding to an homogeneous loaded matrix and to an unloaded inclusion with eigenstrain $\hat{\epsilon}^*$, respectively.

stiffness $\hat{\mathcal{C}}^{(1)}$. We suppose a perfect gluing of the spherical or cylindrical surfaces obtained by means of radial deformations of both bodies. We also suppose that a system of forces remotely applied generates a uniform stress in a homogeneous matrix $\hat{\mathcal{C}}^{(1)}$ (without the inhomogeneity). The corresponding elastic state is fully described by the following fields: linear displacement $u_i^\infty(\vec{y})$, constant strain $\epsilon_{kh}^\infty = \frac{1}{2}\left(\frac{\partial u_k^\infty}{\partial y_h} + \frac{\partial u_h^\infty}{\partial y_k}\right)$ and constant stress $T_{ij}^\infty = \mathcal{C}_{ijkh}^{(1)}\epsilon_{kh}^\infty$. If we now embed the inhomogeneity in the matrix, we must cope with the problem of evaluating the perturbation induced in the elastic fields, both inside and outside the particle. In order to utilize the infinitesimal theory of elasticity we must consider $R_1 \approx R_2$ or, equivalently $|\epsilon_0| \ll 1$ if $\epsilon_0 = R_1/R_2 - 1$. All the quantities in our system are reported in Fig. 2, together with the conceptual scheme utilized to solve the problem. We start with the description of the equivalence principle used to obtain the elastic fields in the system. The original problem with the prestrained (or prestressed) inhomogeneity is approached through the superimposition of two subproblems A and B. The subproblem A is described by an entirely homogeneous matrix subjected to the remote load $\hat{\epsilon}^\infty$ or $\hat{T}^\infty = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^\infty$. In this simple case the following elastic fields apply at any point of the body

$$\hat{\epsilon}^A = \hat{\epsilon}^\infty \quad \text{and} \quad \hat{T}^A = \hat{T}^\infty = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^\infty \tag{16}$$

The subproblem B corresponds to a spherical or cylindrical inclusion (with radius R_1) described by the eigenstrain $\hat{\epsilon}^*$. The results summarized in the previous Section allow us to obtain the following uniform elastic fields in the region of the inclusion

$$\hat{\epsilon}^B = \hat{\mathcal{S}}\hat{\epsilon}^* \quad \text{and} \quad \hat{T}^B = \hat{\mathcal{C}}^{(1)}(\hat{\epsilon}^B - \hat{\epsilon}^*) \tag{17}$$

The superimposition of the stress and the strain for the situations A and B leads to the relations

$$\begin{aligned} \hat{\epsilon} &= \hat{\epsilon}^A + \hat{\epsilon}^B = \hat{\epsilon}^\infty + \hat{\mathcal{S}}\hat{\epsilon}^* \\ \hat{T} &= \hat{T}^A + \hat{T}^B = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^\infty + \hat{\mathcal{C}}^{(1)}(\hat{\epsilon}^B - \hat{\epsilon}^*) \\ &= \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^\infty + \hat{\mathcal{C}}^{(1)}(\hat{\mathcal{S}}\hat{\epsilon}^* - \hat{\epsilon}^*) \end{aligned} \tag{18}$$

which apply to any point of the inclusion. Now, it is important to investigate the relation between $\hat{\epsilon}$ and $\hat{T}$ inside the inhomogeneity, i.e. the constitutive relation of the embedded particle. This is a crucial issue because this relation is described by the stiffness tensor $\hat{\mathcal{C}}^{(2)}$ in the reference frame $\{\vec{x}\}$, where the particle is not deformed (see Fig. 2).

However, this is not true in the reference frame $\{\vec{y}\}$ where the particle is radially deformed in order to achieve the radius R_1 of the cavity. It must be underlined that the equivalence principle must be used with the constitutive equation of the particle written in the reference frame where the particle itself has the same shape and size of the cavity.

The linear displacement field changing the radius of the particle from R_2 to R_1 is $\vec{u}_0(\vec{x}) = (R_1 - R_2)/R_2\vec{x}$ and, therefore, the corresponding strain tensor is $\hat{\epsilon}_0 = (R_1 - R_2)/R_2\hat{I}_2 = \epsilon_0\hat{I}_2$ for a cylinder and $\hat{\epsilon}_0 = (R_1 - R_2)/R_2\hat{I}_3 = \epsilon_0\hat{I}_3$ for a sphere, where $\epsilon_0 = (R_1 - R_2)/R_2$ ($\hat{I}_2$ and $\hat{I}_3$ are the two-dimensional and the three-dimensional identity tensors, respectively).

In other words, each point of the particle is transformed accordingly to $\vec{y} = \vec{x} + \hat{\epsilon}_0\vec{x}$ in order to fit the cavity. In this configuration the surfaces of the prestrained circular inhomogeneity and of the cavity are firmly bonded. An arbitrary deformation $\vec{u}_T(\vec{x})$ of the particle can be described by two successive steps: a first deformation described by $\vec{u}_0(\vec{x})$ and a further deformation $\vec{u}(\vec{y})$ defined on the reference frame $\{\vec{y}\}$. Therefore, the arbitrary deformation $\vec{u}_T(\vec{x})$ can be written in the form $\vec{u}_T(\vec{x}) = \vec{u}_0(\vec{x}) + \vec{u}(\vec{y}) = \vec{u}_0(\vec{x}) + \vec{u}(\vec{x} + \hat{\epsilon}_0\vec{x})$. The subscript T means *true*, i.e. the displacement $\vec{u}_T(\vec{x})$ is the actual or total displacement measured in the reference configuration where the material is in elastic equilibrium. On the other hand, the second step of the deformation, described by the vector $\vec{u}(\vec{y})$, defines a standard strain tensor $\epsilon_{ij}(\vec{y}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i}\right)$ working in the reference frame $\{\vec{y}\}$. The relation between $\hat{\epsilon}_T(\vec{x})$ and $\hat{\epsilon}(\vec{y})$ is

$$\begin{aligned} \epsilon_{T,ij} &= \frac{1}{2}\left(\frac{\partial u_{T,i}}{\partial x_j} + \frac{\partial u_{T,j}}{\partial x_i}\right) \\ &= \frac{1}{2}\left(\frac{\partial u_{0,i}}{\partial x_j} + \frac{\partial u_{0,j}}{\partial x_i} + \frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} + \frac{\partial u_i}{\partial y_s}\epsilon_{0,sj} + \frac{\partial u_j}{\partial y_s}\epsilon_{0,si}\right) \\ &= \epsilon_0\delta_{ij} + (1 + \epsilon_0)\epsilon_{ij} \end{aligned} \tag{19}$$

In the reference frame $\{\vec{x}\}$ we have the standard constitutive equation $\hat{T}(\vec{x}) = \hat{\mathcal{C}}^{(2)}\hat{\epsilon}_T(\vec{x})$ while, in the reference frame $\{\vec{y}\}$ (where the particle has the same radius of the hosting cavity) we simply obtain $\hat{T}(\vec{y}) = \hat{\mathcal{C}}^{(2)}[\epsilon_0\hat{I} + (1 + \epsilon_0)\hat{\epsilon}(\vec{y})]$ where $\hat{I} = \hat{I}_2$ for a cylindrical particle and $\hat{I} = \hat{I}_3$ for a spherical one. By considering that $\epsilon_0 = (R_1 - R_2)/R_2$ we obtain

$$\hat{T}(\vec{y}) = \hat{\mathcal{C}}^{(2)}\left[\frac{R_1 - R_2}{R_2}\hat{I} + \frac{R_1}{R_2}\hat{\epsilon}(\vec{y})\right] = \frac{R_1}{R_2}\hat{\mathcal{C}}^{(2)}\left[\hat{\epsilon}(\vec{y}) - \frac{R_2 - R_1}{R_1}\hat{I}\right] \tag{20}$$

This is the constitutive equation of the prestrained (or prestressed) inhomogeneity in the reference frame $\{\vec{y}\}$. It must be utilized with the strain and stress fields defined in Eq. (18), yielding

$$\underbrace{\hat{\mathcal{C}}^{(1)}\hat{\epsilon}^\infty + \hat{\mathcal{C}}^{(1)}\left(\hat{\mathcal{S}}\hat{\epsilon}^* - \hat{\epsilon}^*\right)}_{\hat{T}(\vec{y})} = \frac{R_1}{R_2}\hat{\mathcal{C}}^{(2)}\left[\underbrace{\left(\hat{\epsilon}^\infty + \hat{\mathcal{S}}\hat{\epsilon}^*\right)}_{\hat{\epsilon}(\vec{y})} - \frac{R_2 - R_1}{R_1}\hat{I}\right] \tag{21}$$

Eq. (21) represents an equation for the eigenstrain $\hat{\epsilon}^*$ assuring the equivalence between the original (prestrained) inhomogeneity problem and the superimposition of the subproblems A and B. The eigenstrain $\hat{\epsilon}^*$ can be obtained through straightforward tensor calculations

$$\begin{aligned} \hat{\epsilon}^* = &\left[\left(\hat{\mathcal{I}} - \frac{R_1}{R_2}(\hat{\mathcal{C}}^{(1)})^{-1}\hat{\mathcal{C}}^{(2)}\right)^{-1} - \hat{\mathcal{S}}\right]^{-1} \\ &\times \left[\hat{\epsilon}^\infty - \left(\hat{\mathcal{I}} - \frac{R_2}{R_1}(\hat{\mathcal{C}}^{(2)})^{-1}\hat{\mathcal{C}}^{(1)}\right)^{-1}\frac{R_2 - R_1}{R_1}\hat{I}\right] \end{aligned} \tag{22}$$

Moreover, Eq. (21) can be written in the alternative form

$$\hat{\mathcal{C}}^{(1)}\left[\underbrace{\left(\hat{\epsilon}^{\infty}+\hat{\mathcal{S}}\hat{\epsilon}^{*}\right)}_{\hat{\epsilon}(\vec{y})}-\hat{\epsilon}^{*}\right]=\frac{R_1}{R_2}\hat{\mathcal{C}}^{(2)}\left[\underbrace{\left(\hat{\epsilon}^{\infty}+\hat{\mathcal{S}}\hat{\epsilon}^{*}\right)}_{\hat{\epsilon}(\vec{y})}-\frac{R_2-R_1}{R_1}\hat{I}\right] \tag{23}$$

which is useful to evaluate the strain $\hat{\epsilon}(\vec{y})$ in the inhomogeneity. A long manipulation leads to the following relation between the internal strain $\hat{\epsilon}(\vec{y})$ and the eigenstrain $\hat{\epsilon}^{*}$

$$\hat{\epsilon}=\left[\hat{\mathcal{I}}-\frac{R_1}{R_2}(\hat{\mathcal{C}}^{(1)})^{-1}\hat{\mathcal{C}}^{(2)}\right]^{-1}\hat{\epsilon}^{*}+\left[\hat{\mathcal{I}}-\frac{R_2}{R_1}\left(\hat{\mathcal{C}}^{(2)}\right)^{-1}\hat{\mathcal{C}}^{(1)}\right]^{-1}\frac{R_2-R_1}{R_1}\hat{I} \tag{24}$$

Now, we can substitute Eq. (22) in Eq. (24), obtaining the internal strain measured in the reference frame $\{\vec{y}\}$

$$\hat{\epsilon}=\hat{\mathcal{A}}\left[\hat{\epsilon}^{\infty}+\hat{\mathcal{S}}(\hat{\mathcal{C}}^{(1)})^{-1}\hat{\mathcal{C}}^{(2)}\frac{R_2-R_1}{R_2}\hat{I}\right] \tag{25}$$

where we have defined the tensor $\hat{\mathcal{A}}$ as

$$\hat{\mathcal{A}}=\left\{\hat{\mathcal{I}}-\hat{\mathcal{S}}\left[\hat{\mathcal{I}}-\frac{R_1}{R_2}(\hat{\mathcal{C}}^{(1)})^{-1}\hat{\mathcal{C}}^{(2)}\right]\right\}^{-1} \tag{26}$$

It is also important to obtain the *true* internal strain, measured in the reference frame $\{\vec{x}\}$. To this aim we obtain from Eq. (19) the relation giving the *true* strain $\hat{\epsilon}_T$ as

$$\hat{\epsilon}_T=\frac{R_1-R_2}{R_2}\hat{I}+\frac{R_1}{R_2}\hat{\epsilon}=\frac{R_1}{R_2}\hat{\mathcal{A}}\left[\hat{\epsilon}^{\infty}-(\hat{\mathcal{I}}-\hat{\mathcal{S}})\frac{R_2-R_1}{R_1}\hat{I}\right] \tag{27}$$

Finally, by recalling the definition of $\hat{\mathcal{A}}$ in Eq. (26), we obtain the explicit expression

$$\hat{\epsilon}_T=\frac{R_1}{R_2}\left\{\hat{\mathcal{I}}-\hat{\mathcal{S}}\left[\hat{\mathcal{I}}-\frac{R_1}{R_2}(\hat{\mathcal{C}}^{(1)})^{-1}\hat{\mathcal{C}}^{(2)}\right]\right\}^{-1}\left[\hat{\epsilon}^{\infty}-(\hat{\mathcal{I}}-\hat{\mathcal{S}})\frac{R_2-R_1}{R_1}\hat{I}\right] \tag{28}$$

This is the most important result of the present Section. It is important to remark that, if we consider $R_1 = R_2$, we obtain the standard Eshelby result for not prestrained inhomogeneities. In fact, if $R_1 = R_2$ both Eqs. (25) and (28) reduce to $\hat{\epsilon}=\hat{\epsilon}_T=\{\hat{\mathcal{I}}-\hat{\mathcal{S}}[\hat{\mathcal{I}}-(\hat{\mathcal{C}}^{(1)})^{-1}\hat{\mathcal{C}}^{(2)}]\}^{-1}\hat{\epsilon}^{\infty}$, as expected. Furthermore, we can calculate the state of strain in the surrounding matrix; the counterpart of Eq. (18) for the external region reads

$$\begin{aligned}\hat{\epsilon}(\vec{y})&=\hat{\epsilon}^{\infty}+\hat{\mathcal{S}}^{\infty}(\vec{y})\hat{\epsilon}^{*}\\ \hat{T}(\vec{y})&=\hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{\infty}+\hat{\mathcal{C}}^{(1)}\hat{\mathcal{S}}^{\infty}(\vec{y})\hat{\epsilon}^{*}\end{aligned} \tag{29}$$

where the eigenstrain $\hat{\epsilon}^{*}$ is given by Eq. (22). The final expression for the external strain assumes the form

$$\hat{\epsilon}(\vec{y})=\hat{\epsilon}^{\infty}+\hat{\mathcal{S}}^{\infty}(\vec{y})\left[\left(\hat{\mathcal{I}}-\frac{R_1}{R_2}\left(\hat{\mathcal{C}}^{(1)}\right)^{-1}\hat{\mathcal{C}}^{(2)}\right)^{-1}-\hat{\mathcal{S}}\right]^{-1}\times\left[\hat{\epsilon}^{\infty}-\left(\hat{\mathcal{I}}-\frac{R_2}{R_1}\left(\hat{\mathcal{C}}^{(2)}\right)^{-1}\hat{\mathcal{C}}^{(1)}\right)^{-1}\frac{R_2-R_1}{R_1}\hat{I}\right] \tag{30}$$

3.1. Formalism for the sphere

Here, we apply the result stated in Eq. (28) to the specific case of a spherical particle. The constitutive equations for the sphere ($j = 2$) and the matrix ($j = 1$) can be represented in the explicit form $\hat{T}=\hat{\mathcal{C}}^{(j)}\hat{\epsilon}=2\mu_j\hat{\epsilon}+\lambda_j\mathrm{Tr}(\hat{\epsilon})\hat{I}_3$. We also introduce the bulk moduli $K_j=\lambda_j+\frac{2}{3}\mu_j$. The explicit expression of the Eshelby tensor for a sphere embedded in a matrix with Poisson ratio ν_1 is reported in literature (Walpole, 1981; Mura, 1987)

$$\mathcal{S}_{ijkh}=\frac{1}{15(1-\nu_1)}[(\delta_{ik}\delta_{jh}+\delta_{ih}\delta_{jk})(4-5\nu_1)+\delta_{kh}\delta_{ij}(5\nu_1-1)] \tag{31}$$

To obtain a more useful form, we can evaluate the effect of $\mathcal{S}_{ijkh}$ over an arbitrary tensor w_{kh}, getting

$$\mathcal{S}_{ijkh}w_{kh}=\frac{2(4-5\nu_1)}{15(1-\nu_1)}w_{ij}+\frac{5\nu_1-1}{15(1-\nu_1)}w_{kk}\delta_{ij} \tag{32}$$

Now, the Poisson ratio ν_1 of the matrix can be written in terms of the moduli K_1 and μ_1 through the standard relation $\nu_1=\frac{3K_1-2\mu_1}{2(3K_1+\mu_1)}$, obtaining

$$\hat{\mathcal{S}}\hat{w}=\frac{6}{5}\frac{K_1+2\mu_1}{3K_1+4\mu_1}\hat{w}+\frac{1}{5}\frac{3K_1-4\mu_1}{3K_1+4\mu_1}\mathrm{Tr}(\hat{w})\hat{I}_3 \tag{33}$$

At this point we have all the ingredients to develop Eq. (28). We define the parameters

$$L_3=1+\frac{6}{5}\frac{K_1+2\mu_1}{3K_1+4\mu_1}\left(\frac{R_1}{R_2}\frac{\mu_2}{\mu_1}-1\right) \tag{34}$$

$$M_3=\frac{1}{5(3K_1+4\mu_1)}\left[5\frac{R_1}{R_2}K_2-K_1\left(3+2\frac{R_1}{R_2}\frac{\mu_2}{\mu_1}\right)-4\left(\frac{R_1}{R_2}\mu_2-\mu_1\right)\right] \tag{35}$$

which are useful to write in explicit form the effect of $\hat{\mathcal{A}}$ defined in Eq. (26) over an arbitrary tensor $\hat{w}$

$$\hat{\mathcal{A}}\hat{w}=\frac{1}{L_3}\hat{w}-\frac{M_3}{L_3}\frac{1}{L_3+3M_3}\mathrm{Tr}(\hat{w})\hat{I}_3 \tag{36}$$

This expression is also useful in the following developments

$$L_3+3M_3=\frac{3\frac{R_1}{R_2}K_2+4\mu_1}{3K_1+4\mu_1} \tag{37}$$

By means of a long but straightforward calculation we obtain the final expression for the *true* strain in the form

$$\hat{\epsilon}_T=\frac{R_1}{R_2}\frac{1}{L_3}\hat{\epsilon}^{\infty}-\frac{R_1}{R_2}\frac{M_3}{L_3}\frac{1}{L_3+3M_3}\mathrm{Tr}\hat{\epsilon}^{\infty}\hat{I}_3-\frac{R_2-R_1}{R_2}\frac{4\mu_1}{3\frac{R_1}{R_2}K_2+4\mu_1}\hat{I}_3 \tag{38}$$

An alternative method for obtaining the value of $\hat{\epsilon}_T$ when $\hat{\epsilon}^{\infty}=0$ can be found in Appendix A. This further calculation has been performed observing that the elastic fields show spherical symmetry when no loads are applied to the system.

3.2. Formalism for the cylinder

We apply now Eq. (28) to the case of a cylindrical particle embedded in the homogeneous matrix. We suppose to deform both the particle and the matrix under the plane strain condition (on the plane perpendicular to the axis of the cylindrical particle). Therefore, we introduce the customarily defined two-dimensional bulk moduli $k_j = K_j + \frac{1}{3}\mu_j (j = 1, 2)$. Accordingly, we adopt the constitutive relations in the form $\hat{T} = \hat{\mathcal{C}}^{(j)}\hat{\epsilon} = 2\mu_j\hat{\epsilon} + (k_j - \mu_j)\mathrm{Tr}(\hat{\epsilon})\hat{I}_2$ where $j = 1$ for the matrix and $j = 2$ for the inhomogeneity. Moreover, we remember that the result of the application of the Eshelby tensor $\hat{\mathcal{S}}$ (for a cylindrical geometry) over an arbitrary tensor $\hat{w}$ is given by (Mura, 1987)

$$\hat{\mathcal{S}}\hat{w} = \frac{1}{2}\frac{k_1 + 2\mu_1}{k_1 + \mu_1}\hat{w} + \frac{1}{4}\frac{k_1 - 2\mu_1}{k_1 + \mu_1}\mathrm{Tr}(\hat{w})\hat{I}_2 \tag{39}$$

The development of Eq. (28) can be made easier by the definition of the parameters

$$L_2 = 1 + \frac{1}{2}\frac{k_1 + 2\mu_1}{k_1 + \mu_1}\left(\frac{R_1}{R_2}\frac{\mu_2}{\mu_1} - 1\right) \tag{40}$$

$$M_2 = \frac{1}{4(k_1 + \mu_1)}\left[2\frac{R_1}{R_2}k_2 - k_1\left(1 + \frac{R_1}{R_2}\frac{\mu_2}{\mu_1}\right) - 2\left(\frac{R_1}{R_2}\mu_2 - \mu_1\right)\right] \tag{41}$$

which are useful to write in explicit form the effect of $\hat{\mathcal{A}}$ defined in Eq. (26) over an arbitrary tensor $\hat{w}$

$$\hat{\mathcal{A}}\hat{w} = \frac{1}{L_2}\hat{w} - \frac{M_2}{L_2}\frac{1}{L_2 + 2M_2}\mathrm{Tr}(\hat{w})\hat{I}_2 \tag{42}$$

The following expression is useful in the calculations

$$L_2 + 2M_2 = \frac{\frac{R_1}{R_2}k_2 + \mu_1}{k_1 + \mu_1} \tag{43}$$

A tedious algebraic manipulation leads to the final result

$$\hat{\epsilon}_T = \frac{R_1}{R_2}\frac{1}{L_2}\hat{\epsilon}^{\infty} - \frac{R_1}{R_2}\frac{M_2}{L_2}\frac{1}{L_2 + 2M_2}\mathrm{Tr}\hat{\epsilon}^{\infty}\hat{I}_2 - \frac{R_2 - R_1}{R_2}\frac{\mu_1}{\frac{R_1}{R_2}k_2 + \mu_1}\hat{I}_2 \tag{44}$$

As before, the value of $\hat{\epsilon}_T$ when $\hat{\epsilon}^{\infty} = 0$ has been checked by a standard methodology based on the cylindrical symmetry of the elastic fields. The details are discussed in Appendix B. Since Eq. (44) is a result obtained for the two-dimensional elasticity (plane strain condition), it can be also verified by means of the complex potentials method of Kolosoff (1909, 1914) and Muskhelishvili (1953). This approach has been followed in Appendix C where the reader can find all the relevant details.

3.3. Example of application

For nano-science applications the typical sizes of the particles range in the interval $5\,\mathrm{nm} < R_1 \approx R_2 < 50\,\mathrm{nm}$, while the possible difference between the radii lies in $0\,Å < |R_2 - R_1| < 5\,Å$. In this Section we show an example of cylindrical particle with moduli $\mu_2 = 85$ GPa and $k_2 = 115$ GPa embedded in a matrix having moduli $\mu_1 = 50$ GPa and $k_1 = 110$ GPa. By means of Eq. (44) we obtain the true internal strain field $\hat{\epsilon}_T$ for $2\,\mathrm{nm} < R_1 < 20\,\mathrm{nm}$ and $0.1\,Å < R_2 - R_1 < 0.5\,Å$. In Fig. 3 (a) one can find the results for the case without external loads applied to the system. In this case the isotropy leads to the hydrostatic condition $\varepsilon_{xx} = \varepsilon_{yy}$. The scale effects generated by the condition $R_1 \neq R_2$ can be compared with the constant value of the strain predicted by the classical Eshelby theory when $R_1 = R_2$. Moreover, in Fig. 3 (b) the effects of a remotely applied uniaxial load are shown. One can observe that the scale effects induced by the condition $R_1 \neq R_2$ become negligible only when the radius R_1 is larger than a given threshold.

We have also analysed the external fields (i.e. outside the particle), described by Eq. (30). In particular, in Fig. 4 the true displacement field components $u_x(x, 0) - u_x^{\infty}(x, 0)$ (a) and $u_y(0, y) - u_y^{\infty}(0, y)$ (b) are shown for $\varepsilon_{xx}^{\infty} = 0.01$, $\varepsilon_{yy}^{\infty} = 0$ and $\varepsilon_{xy}^{\infty} = 0$. We have used the fixed radius $R_1 = 20$ nm and the difference $0.1\,Å < |R_2 - R_1| < 1\,Å$. The dashed lines correspond to the Eshelby theory ($R_1 = R_2$) and, therefore, they are continuous at the cylinder-matrix interface. When $R_1 \neq R_2$ the displacement field shows a discontinuity at the interface due to the gluing of the surfaces having different radius (it is the typical behavior of the elastic fields generated by a dislocation distributed over the interface). The jump of the discontinuity is an increasing function of $R_2 - R_1$ both for the longitudinal and the transversal com-

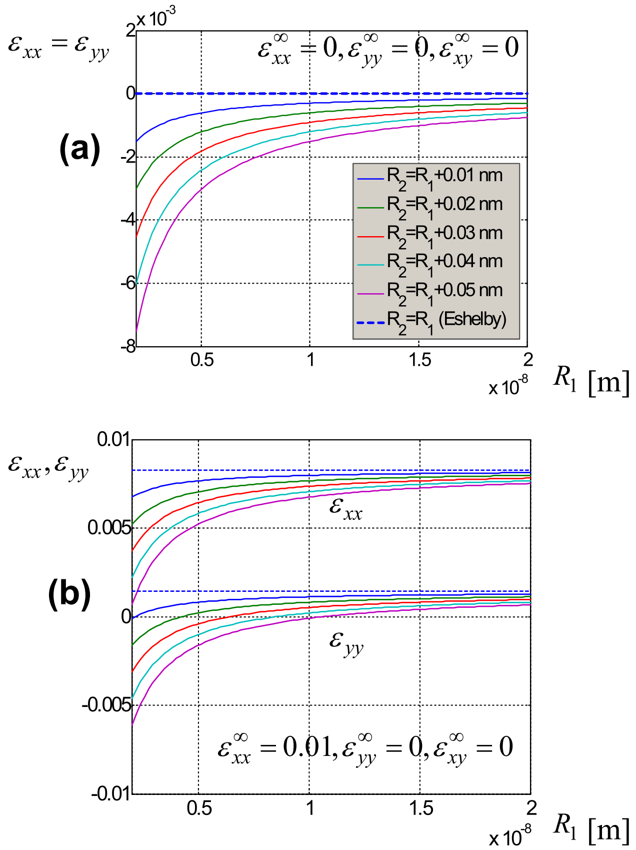

Fig. 3. Internal components ε_{xx} and ε_{yy} of the strain tensor $\hat{\epsilon}_T$ for a prestrained cylindrical inhomogeneity (stiffness $\hat{\mathcal{C}}^{(2)}$ and radius R_2) embedded into a homogeneous matrix (stiffness $\hat{\mathcal{C}}^{(1)}$ and cavity with radius R_1). Results without external load (a) and the effects of a remotely applied deformation (b).

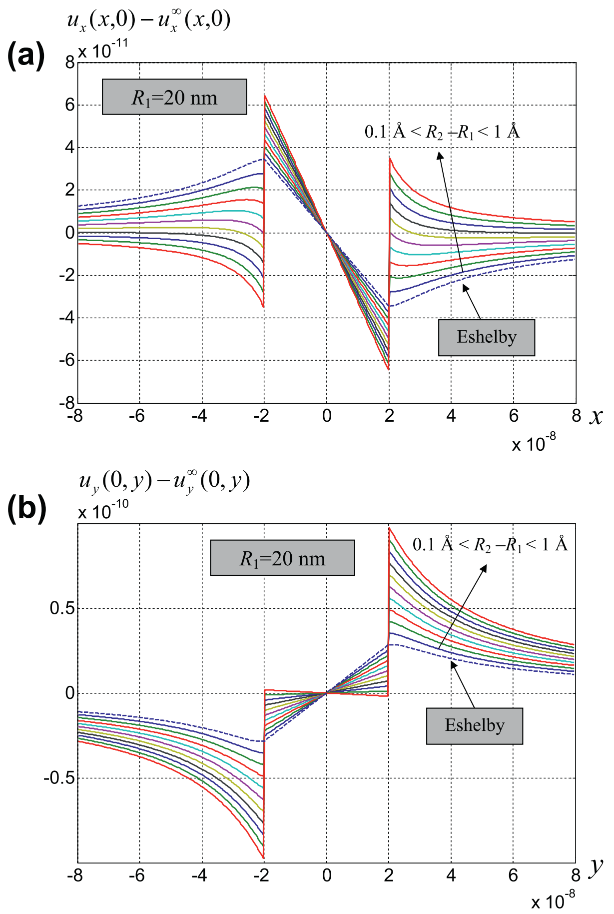

Fig. 4. True displacement field components $u_x(x,0) - u_x^\infty(x,0)$ (a) and $u_y(0,y) - u_y^\infty(0,y)$ (b) for a prestrained cylindrical inhomogeneity (radius R_2) embedded into a homogeneous matrix (cavity with radius R_1).

ponents. By comparing the Eshelby solution with the results for $R_1 \neq R_2$ we note that the behavior can be largely different, depending on the quantity $R_2 - R_1$. As for the longitudinal component, we observe that a value of $R_2 - R_1$ exists (of about 0.7 Å for the example shown in Fig. 4 (a)) which leads to a very fast decay to zero of $u_x(x,0) - u_x^\infty(x,0)$. In other words, in such a case, the prestrain causes a strong localization of the elastic fields around the interface. On the other hand, the transversal component shown in Fig. 4 (b) shows a decay to zero that is more and more slowly for increasing values of $R_2 - R_1$.

All the features described in this Section have been recently confirmed by molecular dynamics experiments conducted in order to show the role of the interface elasticity in nanostructured silicon (Palla et al., 2009, 2010). It is interesting to observe that such atomistic simulations perfectly take into account both the fast decay and the displacement discontinuity, being in good agreement with the present model.

4. Generalized equivalence principle for ellipsoids

As in the case of spherical or cylindrical inhomogeneities, also with ellipsoidal particles we can generalize the Eshelby equivalence principle in order to take into account the effects of the prestrain or prestress. As before, we will prove that the problem can be decomposed in two subproblems: the subproblem A with a loaded uniform matrix and the problem B with an unloaded inclusion with a suitable eigenstrain (see Fig. 5). Here, the new crucial point is the method for obtaining the constitutive equation of the inhomogeneity, written in the configuration fitting the matrix cavity. In fact, in contrast to the previous case where a simple hydrostatic deformation was applied, for an ellipsoidal particle an arbitrary geometrical transformation (rotation and stretching) must be taken into account.

To begin, we consider an elastic particle (stiffness $\hat{C}^{(2)}$) with an ellipsoidal shape given by $\vec{x} \cdot \hat{a}^{-2}\vec{x} = 1$ in the reference frame $\{\vec{x}\}$. This particle must be embedded in the matrix (stiffness $\hat{C}^{(1)}$) showing an ellipsoidal cavity described by $\vec{y} \cdot \hat{b}^{-2}\vec{y} = 1$ in the reference frame $\{\vec{y}\}$ (see Fig. 5 for details). These ellipsoids have the semi-axes aligned to reference frames and, therefore, the tensors $\hat{a}$ and $\hat{b}$ are diagonal and their entries represent the lengths of the semi-axes of the ellipsoids. We now search for the geometrical transformation, which converts the first ellipsoid (representing the particle) in the second ellipsoid (representing the cavity). The general form of such a transformation is assumed in the form $\vec{y} = \hat{F}\vec{x}$ where $\hat{F}$ is an unknown non singular tensor (the inverse transformation is $\vec{x} = \hat{F}^{-1}\vec{y}$). The application of the tensor $\hat{F}$ to the ellipsoid $\vec{x} \cdot \hat{a}^{-2}\vec{x} = 1$ leads to the transformed ellipsoid $\vec{y} \cdot \hat{F}^{-T}\hat{a}^{-2}\hat{F}^{-1}\vec{y} = 1$. Therefore, the tensor $\hat{F}$ must fulfil the condition

$$\hat{b}^{-2} = \hat{F}^{-T}\hat{a}^{-2}\hat{F}^{-1} \tag{45}$$

It is easy to recognize that it exists an infinite number of tensors $\hat{F}$ fulfilling the previous relation: in fact, reasoning in $\Re^n$, the tensor $\hat{F}$ corresponds to n^2 unknowns, while Eq. (45) corresponds to $n(n+1)/2$ equations (in fact, both sides are symmetric). In order to characterize all the possible tensors satisfying Eq. (45) we invoke the polar decomposition $\hat{F} = \hat{R}\hat{U} = \hat{V}\hat{R}$, which applies to any non singular tensor $\hat{F}$. We adopt the left version $\hat{F} = \hat{V}\hat{R}$ where $\hat{R}$ is an orthogonal tensor and $\hat{V}$ is symmetric and positive definite. From the above statement we simply obtain $\hat{F}^{-1} = \hat{R}^T\hat{V}^{-1}$ and $\hat{F}^{-T} = \hat{V}^{-1}\hat{R}$. Therefore, Eq. (45) can be rewritten in the form

$$\hat{b}^{-2} = \hat{V}^{-1}\hat{R}\hat{a}^{-2}\hat{R}^T\hat{V}^{-1} \tag{46}$$

Now, we suppose to consider a given orthogonal tensor $\hat{R}$ and we prove that it exists a unique tensor $\hat{V}$ fulfilling Eq. (46). In other words, we have decomposed the transformation $\vec{y} = \hat{F}\vec{x}$ in two steps: $\vec{z} = \hat{R}\vec{x}$ and $\vec{y} = \hat{V}\vec{z}$ (see Fig. 5 for details). In the reference frame $\{\vec{z}\}$ the ellipsoid assumes the form $\vec{z} \cdot \hat{c}^{-2}\vec{z} = 1$ where $\hat{c} = \hat{R}\hat{a}\hat{R}^T$ is a positive definite symmetric tensor. Since the tensor $\hat{R}$ is now considered fixed, Eq. (46) can be written as follows

$$\hat{b}^{-2} = \hat{V}^{-1}\hat{c}^{-2}\hat{V}^{-1} \tag{47}$$

We must now find the solution $\hat{V}$ of the previous Eq. (47). This equation can be represented in the form $\hat{c}^{-1}\hat{b}^{-2}\hat{c}^{-1} = (\hat{c}^{-1}\hat{V}^{-1}\hat{c}^{-1})(\hat{c}^{-1}\hat{V}^{-1}\hat{c}^{-1})$ or, equivalently, in the form $\hat{c}^{-1}\hat{b}^{-2}\hat{c}^{-1} = (\hat{c}^{-1}\hat{V}^{-1}\hat{c}^{-1})^2$. Therefore, we obtain $\hat{c}^{-1}\hat{V}^{-1}\hat{c}^{-1} = \sqrt{\hat{c}^{-1}\hat{b}^{-2}\hat{c}^{-1}}$ since the tensor $\hat{c}^{-1}\hat{b}^{-2}\hat{c}^{-1}$ is symmetric and positive definite (having a standard square root). At the end, the transformation tensor $\hat{F}^{-1}$ or $\hat{F}^{-T}$ is explicitly given

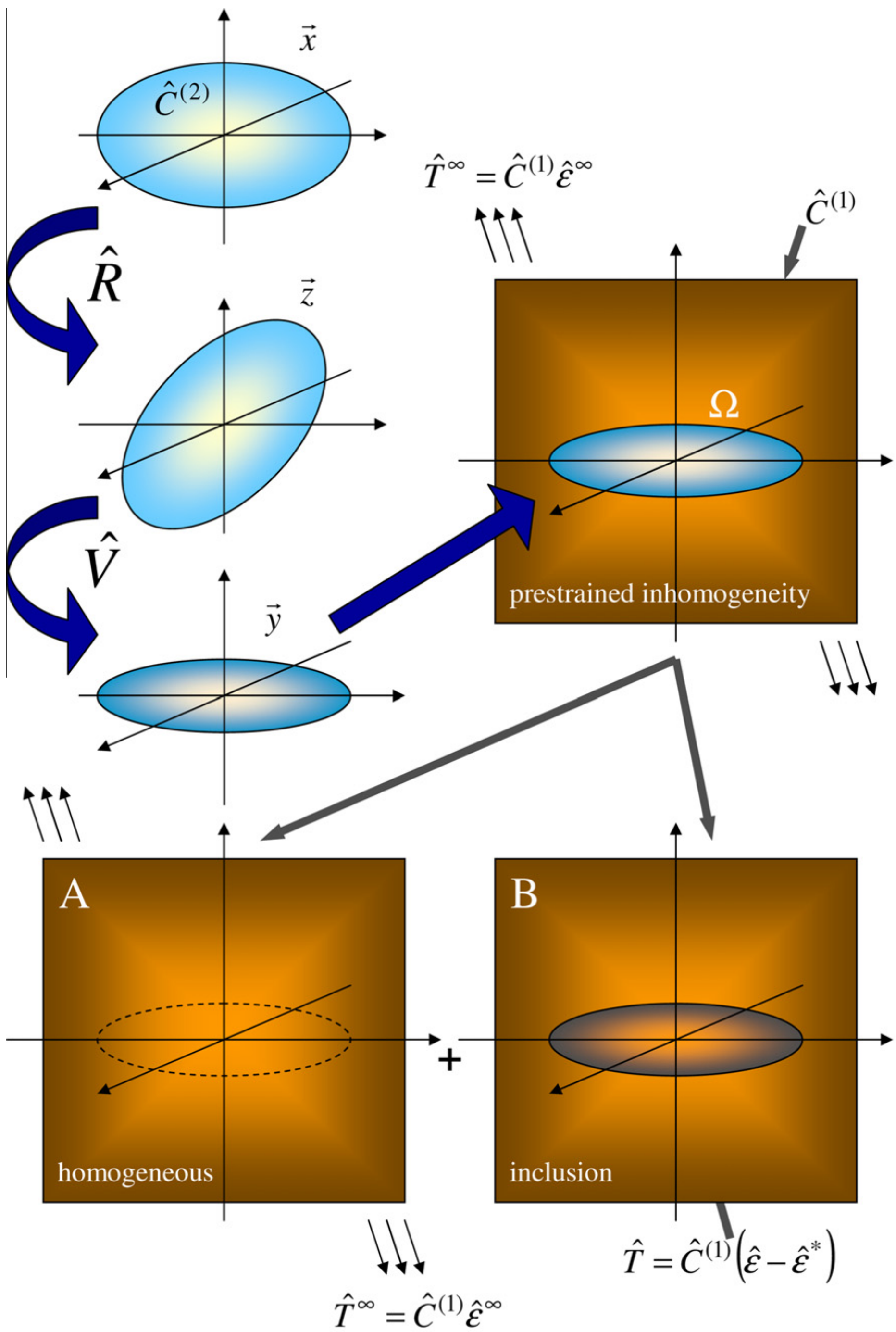

Fig. 5. Scheme of a prestrained ellipsoidal inhomogeneity (stiffness $\hat{C}^{(2)}$) embedded into a matrix (stiffness $\hat{C}^{(1)}$). One can see the initial rotation $\hat{R}$, the further deformation $\hat{V}$ (applied to fit the cavity) and the representation of the generalized equivalence principle: the prestrained ellipsoidal inhomogeneity can be analysed by means of the superimposition of the sub-problems A and B corresponding to an homogeneous loaded matrix and to an unloaded inclusion with eigenstrain $\hat{\epsilon}^{*}$, respectively.

$$\hat{F}^{-1} = \hat{R}^T\hat{V}^{-1} = \hat{R}^T\hat{c}\sqrt{\hat{c}^{-1}\hat{b}^{-2}\hat{c}^{-1}}\hat{c} = \hat{a}\hat{R}^T\sqrt{(\hat{R}\hat{a}\hat{R}^T)^{-1}\hat{b}^{-2}(\hat{R}\hat{a}\hat{R}^T)^{-1}}\hat{R}\hat{a}\hat{R}^T \quad (48)$$

$$\hat{F}^{-T} = \hat{V}^{-1}\hat{R} = \hat{c}\sqrt{\hat{c}^{-1}\hat{b}^{-2}\hat{c}^{-1}}\hat{c}\hat{R} = \hat{R}\hat{a}\hat{R}^T\sqrt{(\hat{R}\hat{a}\hat{R}^T)^{-1}\hat{b}^{-2}(\hat{R}\hat{a}\hat{R}^T)^{-1}}\hat{R}\hat{a} \quad (49)$$

It is simple to verify by substitution that our solution satisfies Eq. (45) as requested. Moreover, if $\hat{R} = \hat{I}_3$ we obtain the simple solution $\hat{F} = \hat{a}^{-1}\hat{b}$ as expected.

For the following purposes we suppose to fix the rotation tensor $\hat{R}$ and to obtain the transformation tensors $\hat{V}$ and $\hat{F}$ through the previous procedure based upon the knowledge of the shape of the ellipsoids (the tensors $\hat{a}$ and $\hat{b}$). Moreover, the shape of the ellipsoid assumed in the reference $\{\vec{z}\}$ must be very similar to that assumed in the reference $\{\vec{y}\}$ in order to satisfy the requirements of the infinitesimal theory of elasticity.

We suppose to measure the *true* strain of the embedded ellipsoid in the reference frame $\{\vec{z}\}$, i.e. after the first rotation. The ellipsoid in the reference frame $\{\vec{z}\}$ assumes the role of reference configuration. In order to describe the generalized version of the Eshelby equivalence principle, the complete transformation, from the reference configuration to the deformed one, can be accomplished in two steps: firstly, we apply the tensor $\hat{V}$, which gives to the ellipsoidal particle the exact shape of the cavity (in the reference frame $\{\vec{y}\}$) and, successively, we consider the final change leading to the actual current configuration.

The transformation $\vec{y} = \hat{V}\vec{z}$ between the reference frames $\{\vec{z}\}$ and $\{\vec{y}\}$ corresponds to a displacement field $\vec{u}_V(\vec{z}) = \vec{y} - \vec{z} = (\hat{V} - \hat{I})\vec{z}$. In this configuration the surfaces of the prestrained ellipsoidal inhomogeneity and of the cavity are firmly bonded. The current configuration, after relaxation, is then reached through a further displacement field $\vec{u}(\vec{y})$, which represents the main unknown in our system, depending upon the shape tensors $\hat{a}$ and $\hat{b}$ and on the externally applied loadings. It is now important to find a relation between the *true* strain measured in the reference frame $\{\vec{z}\}$ and the displacement fields $\vec{u}_V(\vec{z})$ and $\vec{u}(\vec{y})$; the total displacement is $\vec{u}_T(\vec{z}) = \vec{u}_V(\vec{z}) + \vec{u}(\vec{y}) = \vec{u}_V(\vec{z}) + \vec{u}[\vec{z} + \vec{u}_V(\vec{z})]$. The strain in the reference frame $\{\vec{y}\}$ is defined as $\epsilon_{ij}(\vec{y}) = \frac{1}{2}(J_{ij} + J_{ji})$ where $J_{ij}(\vec{y}) = \frac{\partial u_i}{\partial y_j}$ and, therefore, the *true* strain is given by

$$\begin{aligned}\epsilon_{T,ij}(\vec{z}) &= \frac{1}{2}\left(\frac{\partial u_{T,i}}{\partial z_j} + \frac{\partial u_{T,j}}{\partial z_i}\right) \\ &= \frac{1}{2}\left(\frac{\partial u_{V,i}}{\partial z_j} + \frac{\partial u_{V,j}}{\partial z_i} + \frac{\partial u_i}{\partial y_s}\frac{\partial y_s}{\partial z_j} + \frac{\partial u_j}{\partial y_s}\frac{\partial y_s}{\partial z_i}\right) \\ &= \frac{1}{2}[V_{ij} - \delta_{ij} + V_{ji} - \delta_{ji} + J_{is}(\vec{y})V_{sj} \\ &\quad + J_{js}(\vec{y})V_{si}]_{\vec{y}=\hat{V}\vec{z}}\end{aligned} \tag{50}$$

Since the tensor $\hat{V}$ is symmetric we simply obtain

$$\hat{\epsilon}_T(\vec{z}) = \hat{V} - \hat{I} + \frac{1}{2}[\hat{J}(\vec{y})\hat{V} + \hat{V}\hat{J}^{\to T}(\vec{y})]_{\vec{y}=\hat{V}\vec{z}} \tag{51}$$

where $\hat{J}\hat{V}$ and $\hat{V}\hat{J}^{\to T}$ represent two standard matrix multiplications and $\hat{J}^{\to T}$ is the transpose of $\hat{J}$. The constitutive equation in the reference frame $\{\vec{z}\}$ is $\hat{T}(\vec{z}) = \hat{\mathcal{C}}^{(2)}\hat{\epsilon}_T(\vec{z})$ and, consequently, in the reference frame $\{\vec{y}\}$ we immediately obtain

$$\hat{T}(\vec{y}) = \hat{\mathcal{C}}^{(2)}\left\{\hat{V} - \hat{I} + \frac{1}{2}[\hat{J}(\vec{y})\hat{V} + \hat{V}\hat{J}^{\to T}(\vec{y})]\right\} \tag{52}$$

It is important to remark that, in order to adopt the equivalence principle approach, we must utilize in the region Ω the constitutive equation of the inhomogeneity written in the reference frame $\{\vec{y}\}$, i.e. in the deformed configuration close-fitting the cavity of the homogeneous matrix. In these conditions, the problem can be split in the superimposition of two different sub-problems (see Fig. 5): the problem A corresponds to a very simple situation of an entirely homogeneous material (stiffness $\hat{\mathcal{C}}^{(1)}$ without inclusions or inhomogeneities) loaded by the remotely applied stress $\hat{T}^{\infty}$. The corresponding elastic fields can be summed up as follows

$$\hat{\epsilon}^A = \hat{\epsilon}^{\infty}, \hat{J}^A = \hat{\epsilon}^{\infty} \quad \text{and} \quad \hat{T}^A = \hat{T}^{\infty} = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{\infty} \tag{53}$$

The second problem B corresponds to an inclusion confined in the region Ω and described by the eigenstrain $\hat{\epsilon}^*$. The related fields have been discussed in Section 2 and they are summarized below

$$\hat{\epsilon}^B = \hat{\mathcal{S}}\hat{\epsilon}^*, \hat{J}^B = \hat{\mathcal{D}}\hat{\epsilon}^* \quad \text{and} \quad \hat{T}^B = \hat{\mathcal{C}}^{(1)}(\hat{\epsilon}^B - \hat{\epsilon}^*) \tag{54}$$

where the tensors $\hat{\mathcal{D}}$ and $\hat{\mathcal{S}}$ have been defined in Eqs. (14) and (15), respectively. The superimpositions of strain, gradient of displacement and stress in the schemes A and B define the elastic field in the region Ω as follows

$$\begin{aligned}\hat{\epsilon} &= \hat{\epsilon}^A + \hat{\epsilon}^B = \hat{\epsilon}^{\infty} + \hat{\mathcal{S}}\hat{\epsilon}^* \\ \hat{J} &= \hat{J}^A + \hat{J}^B = \hat{\epsilon}^{\infty} + \hat{\mathcal{D}}\hat{\epsilon}^* \\ \hat{T} &= \hat{T}^A + \hat{T}^B = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{\infty} + \hat{\mathcal{C}}^{(1)}(\hat{\epsilon}^B - \hat{\epsilon}^*) = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{\infty} + \hat{\mathcal{C}}^{(1)}(\hat{\mathcal{S}}\hat{\epsilon}^* - \hat{\epsilon}^*)\end{aligned} \tag{55}$$

The equivalence principle becomes operative by combining Eq. (55) for the fields in the region Ω with the constitutive relation given in Eq. (52)

$$\begin{aligned}&\hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{\infty} + \hat{\mathcal{C}}^{(1)}(\hat{\mathcal{S}} - \hat{\mathcal{I}})\hat{\epsilon}^* \\ &\quad = \hat{\mathcal{C}}^{(2)}\left\{\hat{V} - \hat{I} + \frac{1}{2}[(\hat{\epsilon}^{\infty} + \hat{\mathcal{D}}\hat{\epsilon}^*)\hat{V} + \hat{V}(\hat{\epsilon}^{\infty} + \hat{\mathcal{D}}\hat{\epsilon}^*)^{\to T}]\right\}\end{aligned} \tag{56}$$

This is an equation for the eigenstrain ensuring the equivalence between the superimposition of the problems A and B and the original prestrained inhomogeneity problem. This relation can be written in the following form

$$\begin{aligned}&\hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{\infty} - \hat{\mathcal{C}}^{(2)}(\hat{V} - \hat{I}) - \frac{1}{2}\hat{\mathcal{C}}^{(2)}\left(\hat{\epsilon}^{\infty}\hat{V} + \hat{V}\hat{\epsilon}^{\infty}\right) \\ &\quad = -\hat{\mathcal{C}}^{(1)}(\hat{\mathcal{S}} - \hat{\mathcal{I}})\hat{\epsilon}^* + \frac{1}{2}\hat{\mathcal{C}}^{(2)}[(\hat{\mathcal{D}}\hat{\epsilon}^*)\hat{V} + \hat{V}(\hat{\mathcal{D}}\hat{\epsilon}^*)^{\to T}]\end{aligned} \tag{57}$$

which represents a linear equation in the eigenstrain $\hat{\epsilon}^*$. It can be written in components through the standard form $M_{ij} = N_{ijkh}\epsilon^*_{kh}$, where

$$M_{ij} = \mathcal{C}^{(1)}_{ijst}\epsilon^{\infty}_{st} - \mathcal{C}^{(2)}_{ijst}(V_{st} - \delta_{st}) - \frac{1}{2}\mathcal{C}^{(2)}_{ijst}\left(\epsilon^{\infty}_{sk}V_{kt} + V_{sk}\epsilon^{\infty}_{kt}\right) \tag{58}$$

$$N_{ijkh} = \mathcal{C}^{(1)}_{ijkh} - \mathcal{C}^{(1)}_{ijst}\mathcal{S}_{stkh} + \frac{1}{2}\mathcal{C}^{(2)}_{ijst}[\mathcal{D}_{srkh}V_{rt} + \mathcal{D}_{trkh}V_{rs}] \tag{59}$$

Alternatively, Eq. (57) can be solved in tensor notation by means of the definition of the following operation

$$[\hat{\mathcal{D}}\circledast\hat{V}]\hat{\epsilon}^* = \frac{1}{2}[(\hat{\mathcal{D}}\hat{\epsilon}^*)\hat{V} + \hat{V}(\hat{\mathcal{D}}\hat{\epsilon}^*)^T] \tag{60}$$

where the tensor $\hat{\mathcal{D}}\circledast\hat{V}$ corresponds to the components

$$[\hat{\mathcal{D}}\circledast\hat{V}]_{stkh} = \frac{1}{2}[\mathcal{D}_{srkh}V_{rt} + \mathcal{D}_{trkh}V_{rs}] \tag{61}$$

By this definition, Eq. (57) can be easily solved and the equivalent eigenstrain is eventually obtained as

$$\begin{aligned}\hat{\epsilon}^* &= [\hat{\mathcal{C}}^{(2)}(\hat{\mathcal{D}}\circledast\hat{V}) - \hat{\mathcal{C}}^{(1)}(\hat{\mathcal{S}} - \hat{\mathcal{I}})]^{-1} \\ &\quad \times \left[\hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{\infty} - \hat{\mathcal{C}}^{(2)}(\hat{V} - \hat{I}) - \frac{1}{2}\hat{\mathcal{C}}^{(2)}(\hat{\epsilon}^{\infty}\hat{V} + \hat{V}\hat{\epsilon}^{\infty})\right]\end{aligned} \tag{62}$$

Moreover, the *true* internal strain, defined in the reference frame $\{\vec z\}$ is given by Eq. (51). By utilizing Eqs. (55) and (56), it assumes, after some straightforward calculations, the following final form

$$\hat\epsilon_T(\vec z) = (\hat{\mathcal C}^{(2)})^{-1}\hat{\mathcal C}^{(1)}\left\{\hat\epsilon^\infty + (\hat{\mathcal S}-\hat{\mathcal I})\left[\hat{\mathcal C}^{(2)}(\hat{\mathcal D}\circledast\hat V) - \hat{\mathcal C}^{(1)}(\hat{\mathcal S}-\hat{\mathcal I})\right]^{-1} \times\left[\hat{\mathcal C}^{(1)}\hat\epsilon^\infty - \hat{\mathcal C}^{(2)}(\hat V-\hat I) - \frac{1}{2}\hat{\mathcal C}^{(2)}(\hat\epsilon^\infty\hat V + \hat V\hat\epsilon^\infty)\right]\right\} \quad (63)$$

This is the most important result of this Section, stating that the internal strain is uniform inside the inhomogeneity having shape and size different from those of the hosting cavity. We remark that if one is interested in the *true* internal strain in the original reference frame $\{\vec x\}$ it is sufficient to use the rotation $\hat\epsilon_T(\vec x) = \hat R^{-T}\hat\epsilon_T(\vec z)\hat R$. When $\hat V = \hat I$ the inhomogeneity is not strained (deformed) to fit the cavity and the results of the standard Eshelby theory must be obtained. In fact, if $\hat V = \hat I$ we have $\hat{\mathcal D}\circledast\hat V = \hat{\mathcal S}$ and the true strain assumes the simpler form $\hat\epsilon_T = \{\hat{\mathcal I} - \hat{\mathcal S}[\hat{\mathcal I} - (\hat{\mathcal C}^{(1)})^{-1}\hat{\mathcal C}^{(2)}]\}^{-1}\hat\epsilon^\infty$, as expected.

Furthermore, we can calculate the state of strain in the surrounding matrix; the equivalence principle for the external region reads

$$\begin{aligned}\hat\epsilon(\vec y) &= \hat\epsilon^\infty + \hat{\mathcal S}^\infty(\vec y)\hat\epsilon^* \\ \hat T(\vec y) &= \hat{\mathcal C}^{(1)}\hat\epsilon^\infty + \hat{\mathcal C}^{(1)}\hat{\mathcal S}^\infty(\vec y)\hat\epsilon^*\end{aligned} \quad (64)$$

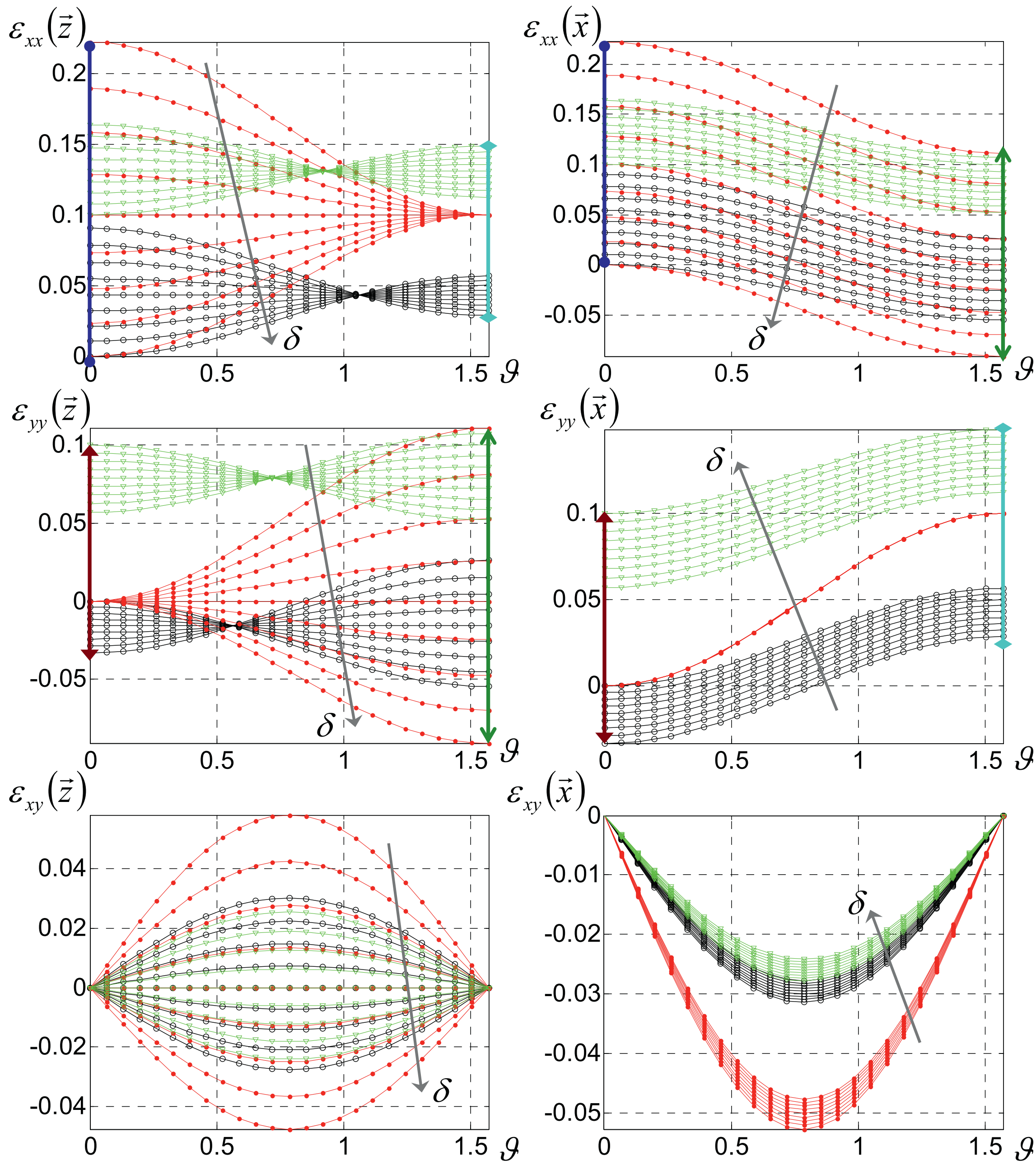

Fig. 6. Planar components ε_{xx}, ε_{yy} and ε_{xy} of the strain $\hat\epsilon_T$ versus the angle ϑ [rad], in both reference frames $\{\vec z\}$ and $\{\vec x\}$ with the following parameters: $K_1 = K_2 = 1$, $\mu_1 = \mu_2 = 0.1$,$\hat a$ = diag(δ,1,1) with $0.9 < \delta < 1.1$ and $\hat b = \mathrm{diag}(1.1, 1, 1)$. The load is given by $\hat\epsilon^\infty = \mathrm{diag}(0.1, 0.1, 0)$. Dotted red lines: prestrain before relaxation; black lines with circles: strain after relaxation without load; green lines with triangles: strain after relaxation with applied load. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

where the eigenstrain $\hat{\epsilon}^*$ is given by Eq. (62). The final expression for the external strain assumes the form

$$\hat{\epsilon}(\vec{y}) = \hat{\epsilon}^\infty + \hat{\mathcal{S}}^\infty(\vec{y})\left[\hat{\mathcal{C}}^{(2)}(\hat{\mathcal{D}}\circledast\hat{V}) - \hat{\mathcal{C}}^{(1)}(\hat{\mathcal{S}} - \hat{\mathcal{I}})\right]^{-1} \times \left[\hat{\mathcal{C}}^{(1)}\hat{\epsilon}^\infty - \hat{\mathcal{C}}^{(2)}(\hat{V} - \hat{I}) - \frac{1}{2}\hat{\mathcal{C}}^{(2)}(\hat{\epsilon}^\infty\hat{V} + \hat{V}\hat{\epsilon}^\infty)\right] \quad (65)$$

We describe now a series of examples of application of the previous theory to prestrained ellipsoidal inhomogeneities inserted into different ellipsoidal cavities. For the sake of simplicity we have used the same material for the embedded particle and the hosting matrix ($K_1 = K_2 = 1$ and $\mu_1 = \mu_2 = 0.1$ in arbitrary units). The geometry of the inhomogeneity is described by the tensor $\hat{a} = \text{diag}(\delta, 1, 1)$ (a.u.) for $0.9 < \delta < 1.1$, in order to investigate the effects of the aspect ratio on the elastic response of the system. More precisely, we have utilized nine values of δ regularly distributed over its range of variation (moving from prolate to oblate ellipsoids of revolution). On the other hand, for the geometry of the cavity we have chosen three different possibilities, namely $\hat{b} = \text{diag}(1.1, 1, 1)$ (a.u.) (prolate spheroid), $\hat{b} = \text{diag}(1, 1, 1)$ (a.u.) (sphere) and $\hat{b} = \text{diag}(0.9, 1, 1)$ (a.u.) (oblate spheroid). For any possible geometry of the ellipsoids, the inhomogeneity is embedded in the cavity

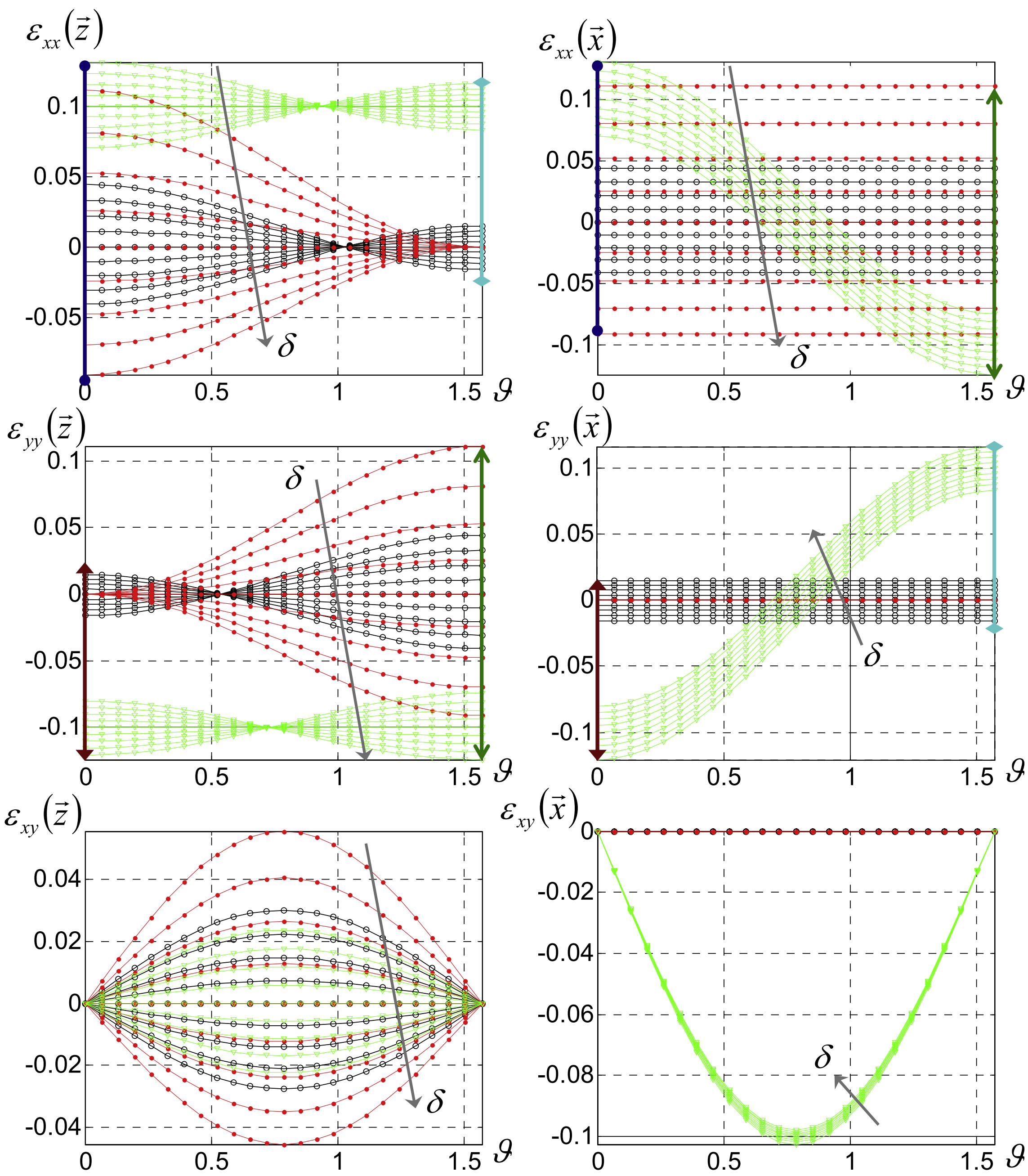

Fig. 7. Planar components ε_{xx}, ε_{yy} and ε_{xy} of the strain $\hat{\epsilon}_T$ versus the angle ϑ [rad], in both reference frames $\{\vec{z}\}$ and $\{\vec{x}\}$ with the following parameters: $K_1 = K_2 = 1$, $\mu_1 = \mu_2 = 0.1$, $\hat{a} = \text{diag}(\delta, 1, 1)$ with $0.9 < \delta < 1.1$ and $\hat{b} = \text{diag}(1, 1, 1)$. The load is given by $\hat{\epsilon}^\infty = \text{diag}(0.1, -0.1, 0)$. Dotted red lines: prestrain before relaxation; black lines with circles: strain after relaxation without load; green lines with triangles: strain after relaxation with applied load. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

after a rotation of an angle ϑ around the $x_3 \equiv z$ axis of the reference frame $\{\vec{x}\}$. It corresponds to a rotation matrix of the form

$$\hat{R} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{66}$$

We have explored the entire interval $0 < \vartheta < \pi/2$ [rad] by means of 25 regularly spaced values. The results have been organized as follows:

- In Fig. 6 the planar components ε_{xx}, ε_{yy} and ε_{xy} of the true strain $\hat{\epsilon}_T$ are shown versus the angle ϑ [rad], in both reference frames $\{\vec{z}\}$ and $\{\vec{x}\}$, for $\hat{b} = \mathrm{diag}(1.1, 1, 1)$. The load is given by $\hat{\epsilon}^\infty = \mathrm{diag}(0.1, 0.1, 0)$.
- In Fig. 7 the planar components ε_{xx}, ε_{yy} and ε_{xy} of the true strain $\hat{\epsilon}_T$ are shown versus the angle ϑ [rad], in both reference frames $\{\vec{z}\}$ and $\{\vec{x}\}$, for $\hat{b} = \mathrm{diag}(1, 1, 1)$. The load is given by $\hat{\epsilon}^\infty = \mathrm{diag}(0.1, -0.1, 0)$.
- In Fig. 8 the planar components ε_{xx}, ε_{yy} and ε_{xy} of the true strain $\hat{\epsilon}_T$ are shown versus the angle ϑ [rad], in both reference frames $\{\vec{z}\}$ and $\{\vec{x}\}$, for $\hat{b} = \mathrm{diag}(0.9, 1, 1)$. The load is given by $\hat{\epsilon}^\infty = \mathrm{diag}(-0.1, -0.1, 0)$.

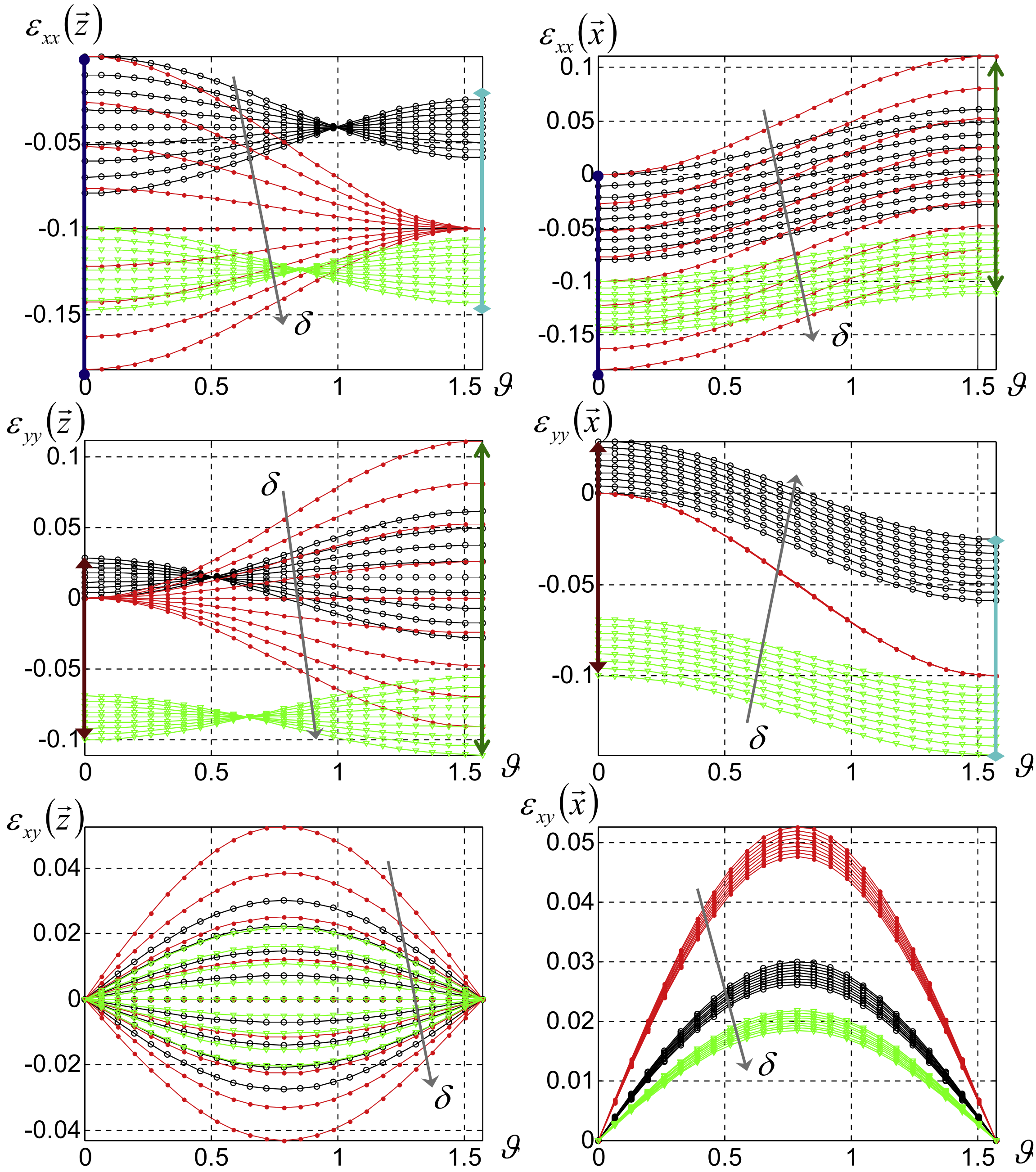

Fig. 8. Planar components ε_{xx}, ε_{yy} and ε_{xy} of the strain $\hat{\epsilon}_T$ versus the angle ϑ [rad], in both reference frames $\{\vec{z}\}$ and $\{\vec{x}\}$ with the following parameters: $K_1 = K_2 = 1$, $\mu_1 = \mu_2 = 0.1$, $\hat{a} = \mathrm{diag}(\delta, 1, 1)$ with $0.9 < \delta < 1.1$ and $\hat{b} = \mathrm{diag}(0.9, 1, 1)$. The load is given by $\hat{\epsilon}^\infty = \mathrm{diag}(-0.1, -0.1, 0)$. Dotted red lines: prestrain before relaxation; black lines with circles: strain after relaxation without load; green lines with triangles: strain after relaxation with applied load. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

In each plot the dotted red lines correspond to the prestrain before relaxation, i.e. $\hat{\epsilon}_T(\vec{z}) = \hat{V} - \hat{I}$ and $\hat{\epsilon}_T(\vec{x}) = \hat{R}^{\to p}\hat{\epsilon}_T(\vec{z})\hat{R} = \hat{R}^{\to p}(\hat{V} - \hat{I})\hat{R}$ (this is the deformation of the inhomogeneity applied for fitting closely the undeformed cavity); the black lines with circles correspond to the strain after relaxation without load, i.e. to Eq. (63) with $\hat{\epsilon}^\infty = 0$ or its rotated version; finally, the green lines with triangles correspond to the strain after relaxation with applied load, i.e. to Eq. (63) with $\hat{\epsilon}^\infty \neq 0$ or its rotated version.

It is interesting to observe the following properties of the plots: for $\vartheta = 0$ we have $\varepsilon_{xx}(\vec{z}) = \varepsilon_{xx}(\vec{x})$ (see dark-blue arrows in Figs. 6–8) and $\varepsilon_{yy}(\vec{z}) = \varepsilon_{yy}(\vec{x})$ (see red arrows in Figs. 6–8); similarly, for $\vartheta = \pi/2$ we have $\varepsilon_{xx}(\vec{z}) = \varepsilon_{yy}(\vec{x})$ (see sky-blue arrows in Figs. 6–8) and $\varepsilon_{yy}(\vec{z}) = \varepsilon_{xx}(\vec{x})$ (see green arrows in Figs. 6–8). These properties simply derive from the rotation of the strain tensor and, therefore, hold on for all the strain plots (unrelaxed, relaxed without load and relaxed with load).

Some more comments of the results follow. In Fig. 6, related to the case with the prolate cavity $\hat{b} = \mathrm{diag}(1.1, 1, 1)$, the unrelaxed strain $\varepsilon_{xx}(\vec{z})$ (dotted red lines) in the reference frame fixed on the matrix must start at the value $(1.1 - \delta)/\delta$ for $\vartheta = 0$ and it must end at the value $(1.1 - 1)/1 = 0.1$ for $\vartheta = \pi/2$; on the other hand, the unrelaxed strain $\varepsilon_{xx}(\vec{x})$ (dotted red lines) in the reference frame fixed on the inhomogeneity must start at the value $(1.1 - \delta)/\delta$ for $\vartheta = 0$ as before while it must end at the value $(1 - \delta)/\delta$ for $\vartheta = \pi/2$. Moreover, for the same case, the unrelaxed strain $\varepsilon_{yy}(\vec{z})$ (dotted red lines) in the reference frame fixed on the matrix must start at the value 0 for $\vartheta = 0$ and it must end at the value $(1 - \delta)/\delta$ for $\vartheta = \pi/2$; on the other hand, the unrelaxed strain $\varepsilon_{yy}(\vec{x})$ (dotted red lines) in the reference frame fixed on the inhomogeneity must start at the value 0 for $\vartheta = 0$ as before while it must end at the value $(1.1 - 1)/1 = 0.1$ for $\vartheta = \pi/2$. These considerations, dealing with the unrelaxed strains, are related just to geometrical factors. On the contrary, the elastic response can be observed in the relaxed strain curves (black lines without load and green lines with load), obtained by means of the present theory, i.e. through Eq. (63). It is interesting to observe that the intersection points of the curves (of $\varepsilon_{xx}(\vec{z})$ for $\vartheta = \pi/2$ and of $\varepsilon_{yy}(\vec{z})$ for $\vartheta = 0$) of the unrelaxed strains (dotted red lines) are shifted by the elastic relaxation process to a different value of the angle ϑ (see black lines), preserving the property that all the curves pass through the same point. This property is maintained also with an externally applied load (see green lines). We also note that a shear strain appear inside the inhomogeneity when it is rotated by an angle ϑ different from 0 and $\pi/2$.

The second case, represented in Fig. 7, is simpler because the cavity is a sphere (of radius 1) and, therefore, the embedding of the inhomogeneity does not depend on the angle ϑ. This can be seen by means of the (dotted red) curves of the unrelaxed strain $\varepsilon_{xx}(\vec{x})$, which are constant at the values $(1 - \delta)/\delta$ and the curves of the unrelaxed strain $\varepsilon_{yy}(\vec{x})$, which are constant at the values 0. Also the relaxed version of these strain curves (black lines without load) are constant for the same reasons. Only when the load is applied to the system we observe the dependence on the angle ϑ, due to the rotation of the reference frame $\{\vec{x}\}$, rigidly bonded to the inhomogeneity (see green lines). As for the reference frame $\{\vec{z}\}$, fixed in the matrix, we observe in Fig. 7 that the red lines for $\varepsilon_{xx}(\vec{z})$ start at $(1 - \delta)/\delta$ and end in 0. Conversely, the red lines for $\varepsilon_{yy}(\vec{z})$ start at 0 and end in $(1 - \delta)/\delta$. The elastic relaxation with or without load allows us to conclude that the intersection points of the curves have the same behavior described for the previous case, represented in Fig. 6.

The third case, shown in Fig. 8 is similar to the first one. Here an oblate particle with $\hat{b} = \mathrm{diag}(0.9, 1, 1)$ is considered and the analysis of the results can be conducted as before: for sake of brevity, it has been left to the reader.

The analysis of the present ellipsoidal case is relevant in the quantum dot applications since, recently, non circular quantum dots have been considered in order to modulate the final quantum response with the aspect ratio of the dot. Since the embedding of an elliptic/ellipsoidal particle is very complicated from the technological point of view, possible deviation of shape and size (between matrix hole and dot) may largely affect the confining properties and the binding energies. Of course, in order to apply the present theory for determining the strain inside the dot it is important to know the exact geometry of cavity and particle before the embedding or the relevant features of the growth process. The elliptical quantum dot has been experimentally analysed in order to investigate the strain induced lateral potential that quantizes the states in the quantum well (Wang et al., 2006). Moreover, the binding energies in ellipsoidal quantum dot are discussed with a variational method by considering the hydrostatic pressure effect. The results show that the binding energies increase with pressure but decrease with increasing ellipticity (Shi and Wei Yan, 2011).

5. Conclusions

The analysis of the mechanical response of nanostructured or nanocomposite materials must be conducted by taking into account the possible slight difference between the embedded particles and their hosting cavities. In fact, at the nanoscale, the lattice mismatch and the geometrical differences between the external surface of each particle and the internal surface of the corresponding cavity can induce important elastic effects. Within this context, the present work has pointed out the following conceptual issues:

- The problem of finding the elastic fields generated by a prestrained or prestressed inhomogeneity (with size and shape slightly different from the hosting cavity) is equivalent to the problem of analysing the effects of a continuous dislocation distributed over the particle/matrix interface. It is a uniform Volterra dislocation for spherical or cylindrical particles and it is a not uniform Somigliana dislocation for arbitrary ellipsoidal inhomogeneities.
- When the inhomogeneity has shape and/or size different from the hosting cavity, in the generalized Eshelby equivalence principle one must consider the constitutive

equation written in the reference frame corresponding to the configuration in which the particle fits exactly the cavity. This approach is useful since it permits the solution of the problems through the inhomogeneity Eshelby theory, avoiding the application of the continuum dislocation theory, which is very complicated for the case of heterogeneous structure.

- We have proved the following geometrical property: given two ellipsoids Υ and Ω defined in two reference frames $\{\vec{x}\}$ and $\{\vec{y}\}$ with axis coincident with their principal axis ($\sum_i x_i^2/a_i^2 = 1$ and $\sum_i y_i^2/b_i^2 = 1$) a unique linear transformation $\vec{y} = \hat{F}\vec{x} = \hat{V}\hat{R}\vec{x}$, which applies Υ in Ω, exists if the rotation matrix $\hat{R}$ is given and fixed ($\hat{V}$ results in a symmetric and positive definite tensor). This property has been used to define the initial strain applied to the ellipsoidal inhomogeneity in order to closely fit the cavity in the undeformed matrix (this is the configuration where the particle–matrix interface is firmly bonded and glued).
- In order to develop the equivalence principle it is necessary to define another version of the Eshelby tensor ($\hat{\mathcal{D}}$) that takes into account the internal gradient of the displacement $\hat{J}$ and not only the strain tensor $\hat{\epsilon}$. From this point of view the standard Eshelby tensor $\hat{\mathcal{S}}$ is simply the symmetrized version of the new tensor $\hat{\mathcal{D}}$.
- When the linear transformation $\vec{y} = \hat{F}\vec{x} = \hat{V}\hat{R}\vec{x}$ is uniform (independent from the position) and the remotely applied strain $\hat{\epsilon}^\infty$ is uniform, then the strain induced inside the prestrained ellipsoidal inhomogeneity remains uniform, similarly to the standard Eshelby theory. We have introduced and discussed a complete procedure able to evaluate such an internal strain and the corresponding external elastic fields for an arbitrary ellipsoidal inhomogeneity embedded in a different ellipsoidal cavity.

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Appendix A. Spherical symmetry

We consider an arbitrary elastic problem with spherical symmetry in a homogeneous and isotropic medium with Lamé moduli λ and μ. We therefore assume that $\vec{u}(\vec{r}) = u(r)\frac{\vec{r}}{r}$, where r is the modulus of the position vector $\vec{r}$. We obtain the general solution for the radial displacement $u(r)$ in static conditions. From the spherically symmetric vector displacement $\vec{u}(\vec{r}) = u(r)\frac{\vec{r}}{r}$ we obtain the components $u_j = \frac{u(r)}{r}x_j$ from which we determine the strain tensor as

$$\epsilon_{ij} = \frac{\partial}{\partial r}\left(\frac{u(r)}{r}\right)\frac{x_i x_j}{r} + \frac{u(r)}{r}\delta_{ij} \tag{A.1}$$

and, therefore, the stress tensor is

$$T_{ij} = 2\mu\frac{\partial}{\partial r}\left(\frac{u(r)}{r}\right)\frac{x_i x_j}{r} + (2\mu + 3\lambda)\frac{u(r)}{r}\delta_{ij} + \lambda r\frac{\partial}{\partial r}\left(\frac{u(r)}{r}\right)\delta_{ij} \tag{A.2}$$

By substituting this form of the stress tensor into the equilibrium equation $\frac{\partial T_{ij}}{\partial x_i} = 0$ we obtain

$$(2\mu + \lambda)\frac{\vec{r}}{r}\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u(r))\right] = 0 \tag{A.3}$$

It follows that the quantity $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)$ must be constant. Therefore, the general solution for the radial displacement is $u(r) = Ar + \frac{B}{r^2}$ (Landau and Lifschitz, 1959; Atkin and Fox, 2005). The corresponding strain tensor is

$$\epsilon_{ij} = \left(A + \frac{B}{r^3}\right)\delta_{ij} - 3B\frac{x_i x_j}{r^5} \tag{A.4}$$

and the radial force (for unit of area) is

$$\vec{f} = \hat{T}\vec{n} = \hat{T}\frac{\vec{r}}{r} = \left[(2\mu + 3\lambda)A - 4\mu B\frac{1}{r^3}\right]\frac{\vec{r}}{r} \tag{A.5}$$

The relations obtained allow us to solve any problem with spherical symmetry. We consider a medium with moduli λ_1 and μ_1 with a spherical cavity of radius R_1. This cavity will be filled by a sphere made of a material having moduli λ_2 and μ_2 and radius R_2 (different from R_1). We suppose to radially deform both media in order to obtain the coincidence of the two spherical surfaces with initial radii R_1 and R_2. We observe that when $R_2 < R_1$ the materials are subjected to radial traction, and, when $R_1 < R_2$ they are subjected to radial compression. We define $u_2(r)$ as the radial displacement in the internal sphere and $u_1(r)$ as the radial displacement in the external matrix. The previous analysis allows us to write the expressions $u_2(r) = Ar + \frac{B}{r^2}$ and $u_1(r) = Cr + \frac{D}{r^2}$. Since we must impose $u_2(0) = 0$ and $u_1(+\infty) = 0$ we assume $B = 0$ and $C = 0$. The perfect gluing between the spherical surfaces imposes the interface relations (continuity of the displacement and of the radial force)

$$R_2 + AR_2 = R_1 + \frac{D}{R_1^2} \quad \text{and} \quad (2\mu_2 + 3\lambda_2)A = -4\mu_1\frac{D}{R_1^3} \tag{A.6}$$

From Eq. (A.6) we find the parameters A and D and, therefore, the solutions of the problem in terms of radial displacements

$$u_2(r) = \frac{R_1 - R_2}{R_1}\frac{1}{\frac{2\mu_2 + 3\lambda_2}{4\mu_1} + \frac{R_2}{R_1}}r \tag{A.7}$$

$$u_1(r) = -\frac{R_1 - R_2}{r^2}R_1^2\frac{1}{1 + \frac{4\mu_1}{2\mu_2 + 3\lambda_2}\frac{R_2}{R_1}} \tag{A.8}$$

By considering that $\lambda_2 = K_2 - \frac{2}{3}\mu_2$ we obtain the equilibrium radius of the gluing

$$R_{eff} = R_2 + u_d(R_2) = R_1 + u_f(R_1) = \frac{3K_2 + 4\mu_1}{\frac{3K_2}{R_2} + \frac{4\mu_1}{R_1}} \tag{A.9}$$

Moreover, the internal (3D) strain is obtained as

$$\hat{\epsilon}_T = \frac{\partial u_2(r)}{\partial r}\hat{I}_3 = \frac{R_1 - R_2}{R_1}\frac{1}{\frac{2\mu_2+3\lambda_2}{4\mu_1}+\frac{R_2}{R_1}}\hat{I}_3$$
$$= \frac{R_1 - R_2}{R_2}\frac{4\mu_1}{3\frac{R_1}{R_2}K_2 + 4\mu_1}\hat{I}_3 \tag{A.10}$$

which is perfectly coherent with Eq. (38) when $\hat{\epsilon}^\infty = 0$ (no external loads applied).

Appendix B. Cylindrical symmetry

We consider an arbitrary elastic problem with cylindrical symmetry in a homogeneous and isotropic medium with Lamé moduli λ and μ. The displacement vector is therefore given by $\vec{u}(\vec{r}) = g(r)\vec{e}_r + f(x_3)\vec{e}_3$, where $r = \sqrt{x_1^2 + x_2^2}$ is the modulus of the vector $(x_1, x_2, 0)$, $\vec{e}_r = (x_1/r, x_2/r, 0)$ and $\vec{e}_3 = (0, 0, 1)$. We determine the general solution for the radial displacement $g(r)$ and for the longitudinal displacement $f(x_3)$. From the displacement vector $\vec{u}(\vec{r}) = g(r)\vec{e}_r + f(x_3)\vec{e}_3$ we simply obtain the strain tensor and the stress tensor; these expressions can be substituted into the equilibrium elasticity equation $\frac{\partial T_{ij}}{\partial x_i} = 0$. The development of the straightforward calculations lead to the differential equations

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rg(r))\right] = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x_3^2} = 0 \tag{B.1}$$

We therefore obtain $g(r) = Ar + \frac{B}{r}$ and $f(x_3) = Cx_3 + D$. The parameter D represents a simple translation: from now on, we assume $D = 0$. The components of the displacement assume the form $u_1 = \left(Ar + \frac{B}{r}\right)\frac{x_1}{r}$, $u_2 = \left(Ar + \frac{B}{r}\right)\frac{x_2}{r}$ and $u_3 = f(x_3) = Cx_3$ (Landau and Lifschitz, 1959; Atkin and Fox, 2005). By differentiation we calculate the strain tensor

$$\hat{\epsilon} = \begin{bmatrix} A + \frac{B}{r^2} - 2B\frac{x_1^2}{r^4} & -2B\frac{x_1x_2}{r^4} & 0 \\ -2B\frac{x_1x_2}{r^4} & A + \frac{B}{r^2} - 2B\frac{x_2^2}{r^4} & 0 \\ 0 & 0 & C \end{bmatrix} \tag{B.2}$$

and the corresponding stress tensor is

$$\hat{T} = 2\mu\begin{bmatrix} A + \frac{B}{r^2} - 2B\frac{x_1^2}{r^4} & -2B\frac{x_1x_2}{r^4} & 0 \\ -2B\frac{x_1x_2}{r^4} & A + \frac{B}{r^2} - 2B\frac{x_2^2}{r^4} & 0 \\ 0 & 0 & C \end{bmatrix} + \lambda(2A + C)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{B.3}$$

Finally, the radial pressure is

$$\hat{T}\vec{e}_r = \left[2(\lambda + \mu)A - 2\mu\frac{B}{r^2} + \lambda C\right]\vec{e}_r \tag{B.4}$$

and the longitudinal pressure is

$$\hat{T}\vec{e}_3 = [(\lambda + 2\mu)C + 2\lambda A]\vec{e}_3 \tag{B.5}$$

These relations can be utilized to solve explicitly any problem of the elasticity theory with cylindrical symmetry. We consider now a material with moduli λ_1 and μ_1 with a cavity of radius R_1 . The cavity will be filled by a cylinder made of a different material having moduli λ_2 and μ_2 and radius R_2. As before, we determine the elastic fields after a perfect gluing of the surfaces. In the internal cylinder we define $g_2(r) = A_2 r + \frac{B_2}{r}$ and $f_2(x_3) = C_2 x_3$. Similarly, in the external matrix we define $g_1(r) = A_1 r + \frac{B_1}{r}$ and $f_1(x_3) = C_1 x_3$. Since $g_2(0) = 0$, we assume $B_2 = 0$; moreover, since $g_1(+\infty) = 0$, we assume $A_1 = 0$; finally, we consider a plane strain condition and, therefore, we set $C_1 = C_2 = 0$. Now, we have to determine the unknowns constants A_2 and B_1 by means of the interface relations

$$R_2 + A_2 R_2 = R_1 + \frac{B_1}{R_1} \quad \text{and} \quad 2(\mu_2 + \lambda_2)A_2 = -2\mu_1\frac{B_1}{R_1^2} \tag{B.6}$$

The final radial displacement can be eventually obtained as

$$g_2(r) = \frac{R_1 - R_2}{R_1}\frac{1}{\frac{\mu_2+\lambda_2}{\mu_1} + \frac{R_2}{R_1}}r$$
$$g_1(r) = -\frac{R_1 - R_2}{r}R_1\frac{1}{1 + \frac{\mu_1}{\mu_2+\lambda_2}\frac{R_2}{R_1}}$$

The radius corresponding to the gluing of the surfaces (at equilibrium) is

$$R_{eff} = R_2 + g_2(R_2) = R_1 + g_1(R_1) = \frac{\lambda_2 + \mu_2 + \mu_1}{\frac{\lambda_2+\mu_2}{R_2} + \frac{\mu_1}{R_1}}$$

By considering that $\lambda_2 = K_2 - \frac{2}{3}\mu_2$ and utilizing the two-dimensional modulus $k_2 = K_2 + \frac{1}{3}\mu_2$ we obtain the internal (2D) strain as

$$\hat{\epsilon}_T = \frac{\partial g_2(r)}{\partial r}\hat{I}_2 = \frac{R_1 - R_2}{R_1}\frac{1}{\frac{\mu_2+\lambda_2}{\mu_1} + \frac{R_2}{R_1}}\hat{I}_2$$
$$= \frac{R_1 - R_2}{R_2}\frac{\mu_1}{\frac{R_1}{R_2}k_2 + \mu_1}\hat{I}_2 \tag{B.7}$$

This result is in perfect agreement with Eq. (44) when $\hat{\epsilon}^\infty = 0$ (no external loads applied).

Appendix C. The two-dimensional problem

In order to solve the model, we use the complex variable method for the two-dimensional elasticity (Atkin and Fox, 2005). In each homogeneous region of the x_1x_2-plane the displacement vector field and the stress tensor field can be represented by means of a couple of Kolossov–Muskhelishvili elastic potentials (Kolosoff, 1909, 1914; Muskhelishvili, 1953). We assume that the elastic state of a given homogeneous region α is exactly described by two holomorphic functions $\phi_\alpha(z)$ and $\psi_\alpha(z)$, where the complex number $z = x_1 + ix_2$ represents the position on the plane. The Kolossov–Muskhelishvili equations allow for the determination of the elastic fields in each region (Muskhelishvili, 1953; Green and Zerna, 1954)

$$u_1^\alpha + i\,u_2^\alpha = \frac{1}{2\mu_\alpha}\left[\chi_\alpha\phi_\alpha(z) - z\,\overline{\phi'_\alpha(z)} - \overline{\psi_\alpha(z)}\right] \tag{C.1}$$

$$T_{11}^\alpha + T_{22}^\alpha = 2\left[\phi'_\alpha(z) + \overline{\phi'_\alpha(z)}\right] \tag{C.2}$$

$$T_{22}^\alpha - T_{11}^\alpha + 2\,i\,T_{12}^{(\alpha)} = 2\left[\bar{z}\phi''_\alpha(z) + \psi''_\alpha(z)\right] \tag{C.3}$$

where $\bar{f}$ is the conjugate of f while f' and f'' indicate the first and the second derivative of the analytic function f, respectively. In our model the phase with $\alpha = 1$ corresponds to the matrix and the phase with $\alpha = 2$ corresponds to the inclusion. It means that $\phi_1(z)$ and $\psi_1(z)$ are defined for $|z| > R_1$ and $\phi_2(z)$ and $\psi_2(z)$ are defined for $|z| < R_2$. Moreover, the parameter χ_α introduced in Eq. (C.1) is given by $\chi_\alpha = 3 - 4\nu_\alpha$ under the assumed plane strain conditions (Muskhelishvili, 1953). The solution of the elastic problem can be obtained by imposing the perfect bonding at the interface described by the following continuity relations

$$\begin{aligned} &(z + u_1^1 + i\, u_2^1)|_{z=R_1e^{i\theta}} = (z + u_1^2 + i\, u_2^2)|_{z=R_2e^{i\theta}} \\ &(\hat{T}^1 \cdot \vec{n})|_{z=R_1e^{i\theta}} = (\hat{T}^2 \cdot \vec{n})|_{z=R_2e^{i\theta}} \end{aligned} \tag{C.4}$$

These boundary conditions can be expressed in terms of the elastic potentials

$$\begin{aligned} &\left(z + \frac{1}{2\mu_1}[\chi_1\phi_1 - z\,\overline{\phi_1'} - \overline{\psi_1}]\right)|_{z=R_1e^{i\theta}} \\ &\quad = \left(z + \frac{1}{2\mu_2}[\chi_2\phi_2 - z\,\overline{\phi_2'} - \overline{\psi_2}]\right)|_{z=R_2e^{i\theta}} \\ &\left(\phi_1 + z\,\overline{\phi_1'} + \overline{\psi_1}\right)|_{z=R_1e^{i\theta}} = \left(\phi_2 + z\,\overline{\phi_2'} + \overline{\psi_2}\right)|_{z=R_2e^{i\theta}} \end{aligned} \tag{C.5}$$

The potentials $\phi_2(z)$ and $\psi_2(z)$ can be represented by Taylor series and $\phi_1(z)$ and $\psi_1(z)$ by Laurent series (Atkin and Fox, 2005; Green and Zerna, 1954). A detailed analysis of the problem proves that the following simplified representations are sufficient to solve the problem

$$\psi_1(z) = \mu_1(\varepsilon_{22}^\infty - \varepsilon_{11}^\infty + 2i\varepsilon_{12}^\infty)z + \frac{H_1}{z} + \frac{H_3}{z^3} \tag{C.6}$$

$$\phi_1(z) = \frac{\mu_1(\varepsilon_{11}^\infty + \varepsilon_{22}^\infty)z}{\chi_1 - 1} + \frac{F}{z} \tag{C.7}$$

$$\psi_2(z) = Az \tag{C.8}$$

$$\phi_2(z) = Bz \tag{C.9}$$

The linear terms in $\phi_1(z)$ and $\psi_1(z)$ represent the remotely applied load described by an arbitrary strain with components $\varepsilon_{11}^\infty, \varepsilon_{22}^\infty$ and ε_{12}^∞. The continuity relations given in Eq. (C.5) lead to a linear system for the complex parameters H_1, H_3, F, A and B. The parameters H_1, H_3 and F describe the elastic fields in the matrix around the inclusion and can be eventually obtained as

$$\begin{aligned} \Re e\{H_1\} = {} & 4\frac{\mu_1\mu_2(R_1 - R_2)R_1^2}{2\mu_2R_1 - \mu_1R_2 + R_2\mu_1\chi_2} \\ & + 2\frac{(\varepsilon_{11}^\infty + \varepsilon_{22}^\infty)[R_1\mu_2(\chi_1 - 1) - R_2\mu_1(\chi_2 - 1)]\mu_1R_1^2}{(2\mu_2R_1 - \mu_1R_2 + R_2\mu_1\chi_2)(\chi_1 - 1)} \end{aligned} \tag{C.10}$$

$$\Im m\{H_1\} = 0 \tag{C.11}$$

$$\Re e\{H_3\} = \frac{R_1^4\mu_1(\varepsilon_{22}^\infty - \varepsilon_{11}^\infty)(\mu_2R_1 - \mu_1R_2)}{R_1\mu_2\chi_1 + \mu_1R_2} \tag{C.12}$$

$$\Im m\{H_3\} = 2\frac{\mu_1R_1^4\varepsilon_{12}^\infty(\mu_1R_2 - \mu_2R_1)}{R_1\mu_2\chi_1 + \mu_1R_2} \tag{C.13}$$

$$\Re e\{F\} = \frac{R_1^2\mu_1(\varepsilon_{22}^\infty - \varepsilon_{11}^\infty)(\mu_2R_1 - \mu_1R_2)}{R_1\mu_2\chi_1 + \mu_1R_2} \tag{C.14}$$

$$\Im m\{F\} = 2\frac{\mu_1R_1^2\varepsilon_{12}^\infty(\mu_1R_2 - \mu_2R_1)}{R_1\mu_2\chi_1 + \mu_1R_2} \tag{C.15}$$

The parameters A and B represent the uniform field in the cylindrical inclusion

$$\Re e\{A\} = \frac{R_1\mu_1\mu_2(\varepsilon_{22}^\infty - \varepsilon_{11}^\infty)(\chi_1 + 1)}{R_1\mu_2\chi_1 + \mu_1R_2} \tag{C.16}$$

$$\Im m\{A\} = 2\frac{\mu_1\mu_2R_1\varepsilon_{12}^\infty(\chi_1 + 1)}{R_1\mu_2\chi_1 + \mu_1R_2} \tag{C.17}$$

$$\begin{aligned} \Re e\{B\} = {} & 2\frac{\mu_1\mu_2(R_1 - R_2)}{2\mu_2R_1 - \mu_1R_2 + R_2\mu_1\chi_2} \\ & + \frac{(\varepsilon_{11}^\infty + \varepsilon_{22}^\infty)(\chi_1 + 1)R_1\mu_2\mu_1}{(2\mu_2R_1 - \mu_1R_2 + R_2\mu_1\chi_2)(\chi_1 - 1)} \end{aligned} \tag{C.18}$$

$$\Im m\{B\} = 0 \tag{C.19}$$

The knowledge of all the parameters allows us to obtain any component of any elastic field by means of the Kolossov–Muskhelishvili Eqs. (C.1), (C.2) and (C.3). It is possible to verify that, if we consider $R_1 = R_2$, we exactly obtain the results of the Eshelby theory for a cylindrical inclusion (Mura, 1987). Our general solution takes into account both the effects of the remotely applied loads and those induced by the different size between the cylinder and the hosting hole (prestrain). We utilize now such a solution to verify Eq. (44) obtained in the main text with a different technique. To this aim, by using Eq. (C.1), we obtain the displacement field inside the particle in the form

$$\begin{aligned} u_1^2 &= \frac{1}{2\mu_2}\Re e\{\chi_2\phi_2(z) - z\,\overline{\phi_2'(z)} - \overline{\psi_2(z)}\} \\ &= \frac{1}{2\mu_2}\{[(\chi_2 - 1)\Re e\{B\} - \Re e\{A\}]x + \Im m\{A\}y\} \end{aligned} \tag{C.20}$$

$$\begin{aligned} u_2^2 &= \frac{1}{2\mu_2}\Im m\left\{\chi_2\phi_2(z) - z\,\overline{\phi_2'(z)} - \overline{\psi_2(z)}\right\} \\ &= \frac{1}{2\mu_2}\{\Im m\{A\}x + [(\chi_2 - 1)\Re e\{B\} + \Re e\{A\}]y\} \end{aligned} \tag{C.21}$$

and, by differentiation, we immediately obtain the corresponding strain tensor

$$\hat{\epsilon}_T = \frac{1}{2\mu_2}\begin{bmatrix} (\chi_2 - 1)\Re e\{B\} - \Re e\{A\} & \Im m\{A\} \\ \Im m\{A\} & (\chi_2 - 1)\Re e\{B\} + \Re e\{A\} \end{bmatrix} \tag{C.22}$$

We successively substitute the relation $\chi_\alpha = 3 - 4\nu_\alpha$, the definition of Poisson ratio $\nu_\alpha = (3K_\alpha - 2\mu_\alpha)/(6K_\alpha + 2\mu_\alpha)$ and the formula $K_\alpha = k_\alpha - \mu_\alpha/3$, by obtaining $\chi_\alpha = 1 + 2\mu_\alpha/k_\alpha$. These conversions can be used in Eqs. (C.16), (C.17), (C.18) and (C.22) proving, after long but straightforward calculations, the perfect agreement between Eqs. (C.22) and (44). It is also possible to verify that the external strain field described by the parameters H_1, H_3 and F corresponds exactly to that calculated in Eq. (30).

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